Exam 1, part 2 – Solutions

1) (a)

$$\sum F = m\ddot{x}$$

$F = mbv^{3/2}$

$$-mbv^{3/2} = m\frac{dv}{dt} \qquad \text{let } \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

$$-bv^{3/2} = \frac{dv}{dx}v$$

$$\int_{x_0}^{x} -dx = \int_{v_0}^{v} \frac{v^{-1/2}}{b}\,dv$$

$$-(x-x_0) = \frac{2}{b}v^{1/2}\Big|_{v_0}^{v}$$

$$-(x-x_0) = \frac{2}{b}\left(\sqrt{v} - \sqrt{v_0}\right)$$

$$v(x) = \left[\sqrt{v_0} - \frac{b}{2}(x-x_0)\right]^2$$

(b) the farthest it gets is when $v = 0$; ie. it stops.

$$0 = \sqrt{v_0} - \frac{b}{2}(x-x_0)$$

so

$$(x-x_0) = \frac{2}{b}\sqrt{v_0}$$

2/ $E = T + U$

$$E = \frac{1}{2}mv^2 + \left(-\frac{x^2}{2} + \frac{x^4}{4}\right)$$

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 - \frac{x^2}{2} + \frac{x^4}{4}\right)$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} - x\dot{x} + x^3\dot{x}$$

but note that $\sum F = m\ddot{x}$

$$+x - x^3 = m\ddot{x}$$

$$\frac{x}{m} - \frac{x^3}{m} = \ddot{x}$$

$$= m\dot{x}\left(\frac{x}{m} - \frac{x^3}{m}\right) - x\dot{x} + x^3\dot{x}$$

$$= \dot{x}x - \dot{x}x^3 - x\dot{x} + x^3\dot{x}$$

$$\frac{dE}{dt} = 0 \quad \checkmark$$

Since energy does not change over time, E is a constant + <u>conserved</u>.

3/(a) determine the force ... does it look like $F = -kx$? when x is a displacement from equilib'm

$$\vec{F} = -\vec{\nabla} U$$

only in x-direction

$$F_x = \frac{-\partial}{\partial x}\left[U_0\left(1-e^{-(x-x_0)/\delta}\right)^2 - U_0\right]$$

$$= -U_0(2)\left(1-e^{-(x-x_0)/\delta}\right)(-1)\left(\frac{-1}{\delta}\right)e^{-(x-x_0)/\delta}$$

$$F_x = \frac{-2U_0}{\delta}\left(1-e^{-(x-x_0)/\delta}\right)e^{-(x-x_0)/\delta}$$

for small separations $(x-x_0) \ll \delta$ use expansion of $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$$F_x = \frac{-2U_0}{\delta}\left[1-\left(1-\frac{(x-x_0)}{\delta}+\frac{(x-x_0)^2}{2!\,\delta^2}+\cdots\right)\right]\left(1-\frac{(x-x_0)}{\delta}+\frac{(\ )^2}{2!\,\delta^2}\cdots\right.$$

$$\boxed{F_x = \frac{-2U_0}{\delta^2}(x-x_0) + \ldots \text{ higher order } (x-x_0) \text{ terms}}$$

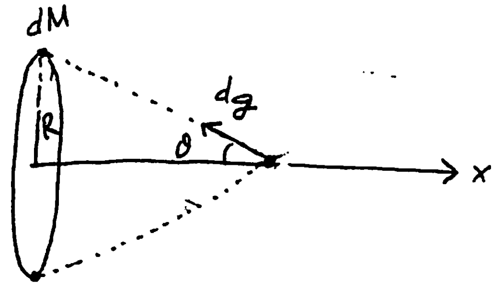
this is of the form $F = -kx$ where $k = \frac{+2U_0}{\delta^2}$

(b) period of oscillation $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{2U_0}{\delta^2 m}}}$

$$T = \pi\delta\sqrt{\frac{2m}{U_0}}$$

4/

(a)



note that only $\hat{x}$ component will be non-zero once all contributions are added

$$g_x = -\int \frac{G\,dM}{r^2}\cos\theta$$

$$= -G\int \frac{dM}{(x^2+R^2)}\,\frac{x}{(x^2+R^2)^{1/2}}$$

note that for every dM, x + R are the same!

$$= \frac{-Gx}{(x^2+R^2)^{3/2}}\int dM$$

$$g_x = \frac{-GMx}{(x^2+R^2)^{3/2}} \Longrightarrow \boxed{\vec{g} = \frac{-GMx}{(x^2+R^2)^{3/2}}\,\hat{x}}$$

(b) Can integrate $\int \vec{g}\cdot d\vec{r} = -\Delta U$ (since $\vec{g} = -\vec{\nabla}U$)

or do the potential directly!

$$\phi = -\int \frac{G\,dM}{r}$$

$$= -\int \frac{G\,dM}{(R^2+x^2)^{1/2}}$$

again, every dM has same R, x.

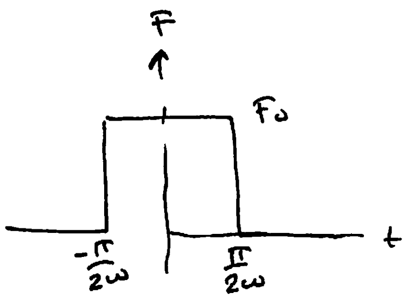
$$\phi = \frac{-GM}{(R^2+x^2)^{1/2}}$$

4/ (c) $\vec{F} = m\vec{g}$

$$= m\left(-\frac{GMx}{(x^2+R^2)^{3/2}}\right)\hat{x}$$

$$\vec{F} = -\frac{GMmx}{(x^2+R^2)^{3/2}}\,\hat{x}$$

5/



$$F(t) = \begin{cases} F_0 & -\frac{\pi}{2\omega} < t < \frac{\pi}{2\omega} \\ 0 & \text{other times in the period} \end{cases}$$

represent this as a Fourier series

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(\omega t)\right]$$

where $a_n = \frac{\omega}{\pi}\int F(t)\cos(n\omega t)\,dt$

$$b_n = \frac{\omega}{\pi}\int F(t)\sin(n\omega t)\,dt$$

here, b_n's $= 0$ since it is an even function. you can show this explicitly also ...

$$b_n = \frac{\omega}{\pi}\int_{-\pi/2\omega}^{+\pi/2\omega} F_0\sin(n\omega t)\,dt$$

$$= \frac{-\omega}{n\omega\pi}F_0\cos(n\omega t)\Big|_{-\frac{\pi}{2\omega}}^{+\frac{\pi}{2\omega}}$$

$$b_n = \frac{-F_0}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\left(-\frac{n\pi}{2}\right)\right]$$ now, $\cos(\theta) = \cos(-\theta)$

$$b_n = \frac{-F_0}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right] = 0$$

try the a's ...

$$a_0 = \frac{\omega}{\pi}\int_{-\frac{\pi}{2\omega}}^{+\pi/2\omega} F_0\,dt = \frac{\omega}{\pi} F_0 t \Big|_{-\pi/2\omega}^{+\pi/2\omega} = \frac{\omega}{\pi} F_0 \frac{\pi}{\omega} = F_0$$

$$a_n = \frac{\omega}{\pi}\int_{-\frac{\pi}{2\omega}}^{+\pi/2\omega} F_0 \cos(n\omega t)\,dt$$

$$= \frac{\omega F_0}{\pi n \omega} \sin(n\omega t)\Big|_{-\frac{\pi}{2\omega}}^{\pi/2\omega}$$

$$= \frac{F_0}{\pi n}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right]$$ now $\sin(-\theta) = -\sin(\theta)$

$$a_n = \frac{2F_0}{\pi n}\sin\left(\frac{n\pi}{2}\right)$$ zero when n is even; $n = \pm 1$ when n is odd

for $n=1$ $\quad a_1 = \frac{2F_0}{\pi}$

$n=3$ $\quad a_3 = -\frac{2F_0}{3\pi}$

$$\boxed{F(t) = \frac{1}{2}F_0 + \frac{2F_0}{\pi}\cos(\omega t) - \frac{2}{3}\frac{F_0}{\pi}\cos(3\omega t) + \ldots - \ldots}$$