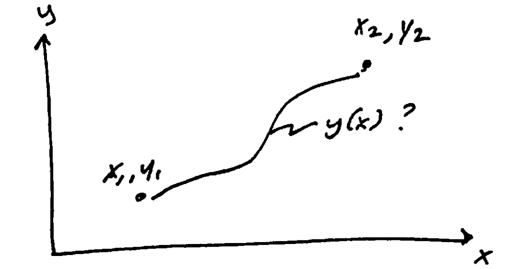
1/



$$\text{distance} = \int ds$$

$$= \int \sqrt{dx^2 + dy^2}$$

$$d. = \int_{x_1}^{x_2} dx \underbrace{\sqrt{1+y'^2}}_{f(y,y',x)}$$

according to the calculus of variations, this has an extremum when

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$0 - \frac{d}{dx}\left(\frac{1}{2}(1+y'^2)^{-1/2}(2y')\right) = 0$$

or that $$\frac{y'}{(1+y'^2)^{1/2}} = \text{constant}, \sqrt{a}$$

$$y'^2 = a(1+y'^2)$$

$$y'^2 - ay'^2 = a$$

$$y'^2 = \frac{1}{1-a}$$

$$y' = (1-a)^{-1/2}$$, also a constant. let $c = (1-a)^{1/2}$

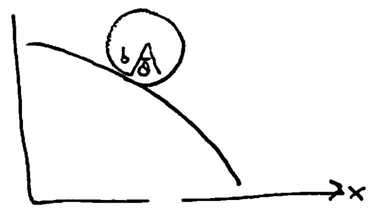
$$\frac{dy}{dx} = c$$

$$y = cx + b, \ldots$$ the equat for a line.

2/ see page 230, last ¶.
alternatively, see pg 237. (used $L = T - U$)

3/ see pg. 254-255. The derivation starts from eq. 7.96 → 7.98.

4/ (a) the rolling disk could be described by 3 coordinates.. 2 for position (x, y)
1 for rotation (θ)



However, x & y are related..

$$y = y_0 - ax^2$$

$$y - y_0 + ax^2 = 0 \qquad (1)$$

And θ is also related to x&y... $ds = b\,d\theta$ (from $s = R\theta$)

but it rolls w/o slipping.. $\sqrt{dy^2 + dx^2} = b\,d\theta \qquad (2a)$

from the first equation(1) $dy - 2ax\,dx = 0$

so (2) becomes

$$\sqrt{4a^2x^2\,dx^2 + dx^2} = b\,d\theta$$

$$dx\sqrt{4a^2x^2 + 1} = b\,d\theta$$

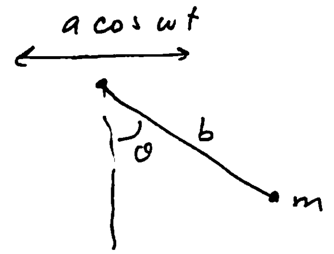
$$\cancel{b\,d\theta - dx\sqrt{4ax^2+1} = 0} \qquad (2b)$$

(b) if you substitute these (1, 2a) into the Lagrangian, only one coordinate is necessary... either x or θ. (3 coordinates − 2 eqns = 1)

(c) θ or x, as explained above

(d) (2b) already is

5/



$$x = a\cos\omega t + b\sin\theta$$

$$\dot{x} = -\omega a\sin\omega t + b\dot{\theta}\cos\theta$$

$$y = -b\cos\theta$$

$$\dot{y} = \dot{\theta}\, b\sin\theta$$

$$L = T - U$$

$$= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \tfrac{1}{2}m\left[(-\omega a\sin\omega t + b\dot{\theta}\cos\theta)^2 + (\dot{\theta}\, b\sin\theta)^2\right] + mgb\cos\theta$$

$$= \tfrac{1}{2}m\left[\omega^2 a^2\sin^2\omega t + b^2\dot{\theta}^2\cos^2\theta - 2a\omega b\dot{\theta}\cos\theta\sin\omega t + \dot{\theta}^2 b^2\sin^2\theta\right] + mgb\cos\theta$$

$$L = \tfrac{1}{2}m\left[b^2\dot{\theta}^2 - 2\omega ab\,\dot{\theta}\cos\theta\sin\omega t + \omega^2 a^2\sin^2\omega t\right] + mgb\cos\theta$$

only one coordinate .. θ

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$0 = \left[mgb(-\sin\theta) + \tfrac{1}{2}m(-2\omega ab\dot{\theta}\sin\omega t)(-\sin\theta)\right]$$

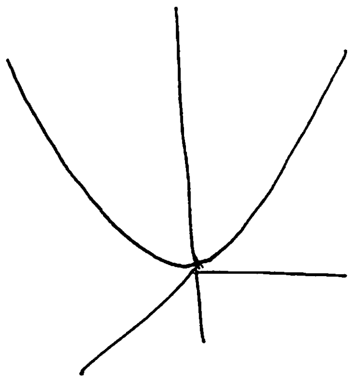
$$- \frac{d}{dt}\left[\tfrac{1}{2}m\left(2b^2\dot{\theta} - 2\omega ab\cos\theta\sin\omega t\right)\right]$$

$$0 = -gb\sin\theta + \omega ab\,\dot{\theta}\sin\theta\sin\omega t - b^2\ddot{\theta} - (-\omega ab\sin\omega t)(-\sin\theta)\dot{\theta}$$

$$- (-\omega ab\cos\theta)(\omega\cos\omega t)$$

$$\boxed{0 = \ddot{\theta} + \frac{g}{b}\sin\theta - \frac{\omega^2 a}{b}\cos\theta\cos\omega t}$$

4/



use cylindrical coordinates

ρ, θ, z

here $\rho^2 = a z^2$

$\rho = \sqrt{a}\, z \Rightarrow \dot\rho = \sqrt{a}\,\dot z \Rightarrow \dot\rho^2 = a\dot z^2$

$$L = T - U$$
$$= \frac{1}{2} m\left(\dot\rho^2 + \rho^2\dot\theta^2 + \dot z^2\right) - mgz$$
$$= \frac{1}{2} m\left(a\dot z^2 + a z^2\dot\theta^2 + \dot z^2\right) - mgz$$
$$= \frac{1}{2} m\left(a z^2\dot\theta^2 + (1+a)\dot z^2\right) - mgz$$

there are 2 coordinates, z + θ

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0$$

$$0 - \frac{d}{dt}\,\frac{1}{2}m(2az^2)(\dot\theta) = 0$$

so $\boxed{maz^2\dot\theta = \text{constant.}}$

the momentum $p_\theta = \dfrac{\partial L}{\partial \dot\theta} = maz^2\dot\theta$

is a constant since

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0.$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot z} = 0$$

$$0 = -mg + \frac{1}{2}ma\dot\theta^2(2z) - \frac{d}{dt}\left[\frac{1}{2}m(1+a)(2\dot z)\right]$$

$$0 = -mg + maz\dot\theta^2 - m(1+a)\ddot z$$

$$\boxed{0 = (1+a)\ddot z - az\dot\theta^2 + g}$$

here, $p_z = \dfrac{\partial L}{\partial \dot z} = m(1+a)\dot z$

is not a constant since

$$\frac{\partial L}{\partial z} - \frac{d}{dt}p_z = 0$$

$$\dot p_z = \frac{\partial L}{\partial z} \neq 0.$$

7/(a) $H = \sum p_i \dot{q}_i - L$ where $L = T - U$

here $U = -\int F_r \, dr$

$$= -\int -k r^{-3} e^{-\beta t} dr = +k e^{-\beta t} \int r^{-3} dr = k e^{-\beta t} \left(-\frac{1}{2}\right) r^{-2}$$

$$U = -\frac{k e^{-\beta t}}{2r^2}$$

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$ to account for 2 dimensions.

so $L = T - U$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{2r^2} e^{-\beta t}$$

the momenta are...

$$p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$\dot{r} = \frac{p_r}{m} \qquad \dot{\theta} = \frac{p_\theta}{m r^2}$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$= p_r \frac{p_r}{m} + \frac{p_\theta p_\theta}{m r^2} - \frac{1}{2} m \left(\frac{p_r^2}{2m} + r^2 \frac{p_\theta^2}{m^2 r^4} \right) - \frac{k}{2r^2} e^{-\beta t}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2 m r^2} - \frac{k}{2r^2} e^{-\beta t}$$

(b) $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{2p_\theta}{2mr^2} = \boxed{\frac{p_\theta}{mr^2} = \dot{\theta}}$$

$$\boxed{\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = 0}$$ so p_θ is a conserved quantity

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p}_r = \frac{-\partial H}{\partial r}$$

$$= -\left(\frac{p_\theta^2}{2m}(-2r^{-3}) - \frac{k}{2}(-2r^{-3})e^{-\beta t}\right)$$

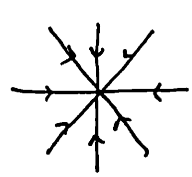
$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{k}{r^3}e^{-\beta t}$$

(c) The Hamiltonian is the total energy. By inspection,

$$H = T + U = E$$

It is not conserved ... there's an explicit time dependence. E decreases.

(d) p_θ is conserved because the force & therefore the Lagrangian is symmetric (invariant) with respect to rotations centered about the origin.



$\vec{F}$... any rotation about origin will return same $\vec{F}$.

p_r is not conserved because the force & therefore the Lagrangian is not symmetric (or invariant) with respect to translations. That is,

$$L(\vec{r}) \neq L(\vec{r} + \delta\vec{r})$$