2/22 In Class – An Integration Primer

As part of calculating the electric fields for continuous charge distributions (solid objects with charge distributed over them), we need to integrate over lines, surfaces and volumes. Here's a primer on how to do that.

Recall that for each little tiny piece of the object dq, we will pretend it's a point charge. Then

$$d\vec{E} = \frac{K dq}{r^2}\hat{r}$$

where $\vec{r}$ goes from dq to the point of interest.

Recall that we're dividing the object into tiny pieces of charge we are calling dq. Then we have

$$\begin{aligned} dq &= \lambda d\ell \\ dq &= \sigma da \\ dq &= \rho dV \end{aligned}$$

where
λ is the charge per unit length and $d\ell$ is a tiny piece of that length
σ is the charge per unit area and da is a tiny piece of that area
ρ is the charge per unit volume and dV is a tiny piece of that volume

For now, no worries about the charge. Let's just focus on how to get and use $d\ell$, da, and dV.

A quick (or not so quick) test of your $d\ell$ is that if you integrate over it, you should get the length of your object.

1. Find the length of a rod of length L by direct integration over $d\ell$. I would probably do this along the x axis, so $d\ell = dx$. (It's not really necessary for this, but it might help set up the idea of translating $d\ell$, da, and dV into coordinates.) Big hint: $L = \int_0^L dx$. Draw a picture and label a dx.

2. Check $C = \int d\ell$ for a ring of radius R, where C is the circumference of a circle. This time you figure out $d\ell$ and the limits of the integral. Draw a picture and label a $d\ell$. Hint: $d\ell$ is an arclength.

3. Repeat the last problem for a semi-circle instead of a ring. What changes?

4. Check $A = \int da$ for a rectangle of height h and width w. Do it two ways as outlined below:

(a) Assume the height is fixed at h, and integrate over dx. Thus, $da = hdx$. Draw a picture with a shaded rectangle for an example hdx. (See my figure in the next problem for an example of a da.) Integrate. What are the limits of your integral? Did you get what you expected?

(b) This time, assume the width is fixed at w, and integrate over dy. Thus, $da = wdy$. Draw a picture with a shaded rectangle for an example wdy. Integrate. What are the limits of your integral? Did you get what you expected?

5. Check $A = \int da$ for the triangle shown below. The triangle is a right triangle and has height h and base b. The following steps will guide you:

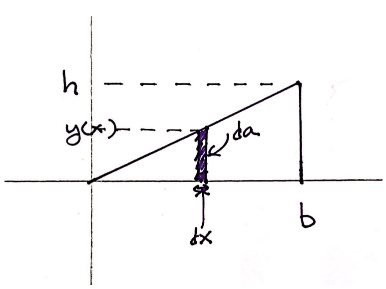
   (a) I have drawn a small da for you. For that da, you would say $da = ydx$. Why can't you use hdx as you did in the last problem?

   (b) In this case, y depends on x, and you can't do the integral over x till you write $y(x)$. Do that. Write $y(x)$ in terms of the constants b and h.

   (c) Substitute your expression for $y(x)$ into $\int ydx$.

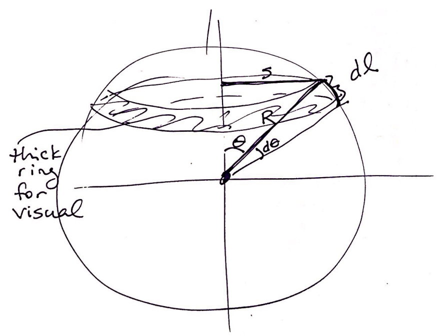
   (d) What are the limits of your integral? Integrate.

   (e) Did you get what you expected?



6. You can use the circumference of circle, $2\pi r$, and integrate that over dr to get the area of a circle, or disk. Assume the disk has a radius of R. Try it. Draw a picture and label da. Clearly show the limits of your integral. Did you get what you expected?

7. You can also use the circumference of a ring to sweep out the surface of a sphere. I have made a sketch of a sphere of radius R. If you have never seen spherical coordinates this might be a challenge. I'll try to guide you:



(a) We need to find a da that will work for the surface of sphere. One way to write this would be $da = 2\pi s d\ell$ where s is the radius of the ring and $d\ell$ is a tiny arclength along the surface of the sphere. In the figure below, I have labeled the radius *of the sphere* R, *of the ring*, s, and θ. Can you find a general expression for the radius *of the ring* s in terms of R and θ? (Notice that R goes from the origin to a point on the ring, and so is not the radius of the ring.)

(b) Now, recalling arclength, can you find the section of $d\ell$ that would be along the surface of the sphere as you move the ring down?

(c) Plug your expressions for s and $d\ell$ into your integral. What are you integrating over? What are the limits?

(d) Integrate. Do you get the answer you expect?

8. Just like we used lengths to build areas, we can use areas to build volumes. In the last problem, you should have found that the surface area of a sphere is $4\pi r^2$. You can integrate to find volume with a $dV = 4\pi r^2 dr$. Try it for a sphere of radius R. Do you get what you expect?

9. Use the area of a disk to sweep out the volume of a cone with base of radius R and height h. Draw your dV. Try it.