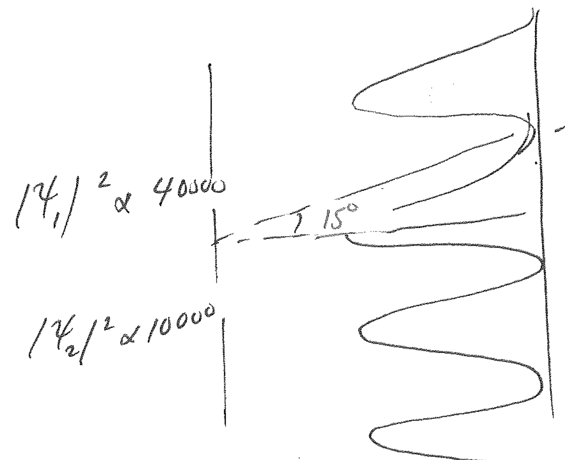
Exam 2 - Solutions

1/



here, $|\Psi_T|^2 = (\Psi_1^2 + \Psi_2^2 + 2\Psi_1\Psi_2\cos\theta)$

θ: phase shift between the 2 waves

at maxima $\theta = 0, 2\pi, 4\pi \ldots$

minima $\theta = \pi, 3\pi, 5\pi$

$$|\Psi_T|^2 = \left[4\cdot10^4 + 1\cdot10^4 + 2(2\cdot10^2)(1\cdot10^2)(-1)\right]$$

$$= 1\cdot10^4$$

* there are 2 different angles here
  1. where the minima/maxima are
  2. shift between the waves

2/

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x \geq \frac{\hbar}{2m\Delta v}$$

$$\Delta x \geq \frac{6.63\cdot10^{-34}/2\pi}{2(9.1\cdot10^{-31})(2\cdot10^5)}$$

$$\Delta x \geq 0.29\text{nm}$$

Position measurements would have a distribution... can be described by an average $(\bar{x})$ and a std dev. The std. dev. in position would be ≥ 0.29nm.

3/ (a) $\lambda = \frac{h}{p}$ and $E = \frac{p^2}{2m} \rightarrow p = \sqrt{2mE}$

$$\lambda = \frac{h}{\sqrt{2mE}}$$

$$\lambda = \frac{1.05 \cdot 10^{-34}}{\left[2(9.1 \cdot 10^{-31})(2 \cdot 10^{3})(1.6 \cdot 10^{-19})\right]^{1/2}}$$

(b) diffraction pattern would be easily seen if slit width were of the same order as λ.

If slit width $\gg \lambda$, won't see the pattern.

4/ emitted light λ_t has energy $E = \frac{hc}{\lambda}$

$\frac{hc}{\lambda_t} = \Delta E = E_2 - E_1$ where $E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2}$

$$\frac{hc}{\lambda_t} = \frac{(2)^2 \hbar^2 \pi^2}{2mL^2} - \frac{(1)^2 \hbar^2 \pi^2}{2mL^2}$$

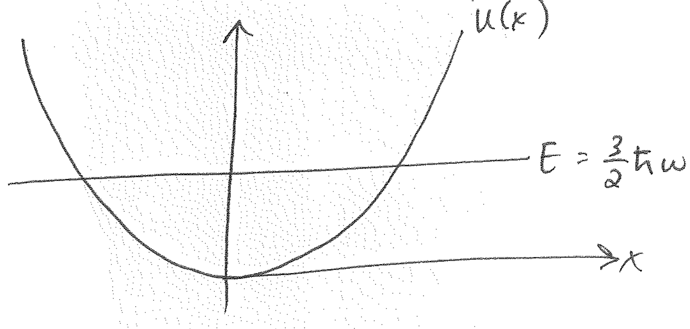
$$\frac{hc}{\lambda_t} = \frac{3 \hbar^2 \pi^2}{2mL^2}$$

$$L = \sqrt{\frac{3 \hbar^2 \pi^2 \lambda_t}{2m\,hc}}$$

$$\frac{\hbar^2}{h} = \frac{\hbar^2}{2\pi \hbar} = \frac{\hbar}{2\pi}$$

$$L = \sqrt{\frac{3 \hbar \pi \lambda_t}{4mc}}$$

5/



correction

$\psi(x) = C x e^{-\frac{1}{2}b^2x^2}$

the classically bound region is defined by where $E = U$.

$$E = U$$

$$\frac{3}{2}\hbar\omega = \frac{1}{2}m\omega^2x^2$$

$$x^2 = \frac{3\hbar\omega}{m\omega^2}$$

$$x = \pm\sqrt{\frac{3\hbar}{m\omega}}$$

$$P = \int_{x_i}^{x_f} |\Psi|^2 dx$$

$$= \int_{-\sqrt{\frac{3\hbar}{m\omega}}}^{+\sqrt{\frac{3\hbar}{m\omega}}} C^2x^2e^{-b^2x^2}\,dx$$

$$P = 2C^2\int_0^{+\sqrt{\frac{3\hbar}{m\omega}}} x^2e^{-b^2x^2}\,dx$$

6/

(a) A is the normalization constant

$$1 = \int_{\text{all space}} |\Psi|^2 \, dx \qquad \text{when} \quad |\Psi|^2 = \Psi^* \Psi$$

$$1 = \int_0^\infty A x e^{-3x} e^{+iEt/\hbar} \, A x e^{-3x} e^{-iEt/\hbar} \, dx$$

$$1 = A^2 \int_0^\infty x^2 e^{-6x} \, dx \qquad \text{when} \quad \int_0^\infty x^m e^{-bx} \, dx = \frac{m!}{b^{m+1}}$$

here $m = 2$, $b = 6$

$$\frac{2!}{6^3}$$

$$1 = A^2 \frac{2}{6^3}$$

$$A = \frac{6^3}{2}$$

(b) most likely is when prob. density ($= |\Psi|^2$) is a maximum.

to find max..

$$\left. \frac{d(\text{prob. density})}{dx} \right|_{x_{peak}} = 0$$

$$\frac{d}{dx}\left(A^2 x^2 e^{-6x}\right) = A^2 \left[2x e^{-6x} + x^2 (-6) e^{-6x} \right]$$

$$= A^2 e^{-6x} x \left[2 - 6x \right] = 0$$

$$x = 0, \infty, \frac{1}{3}$$

$\Psi \to 0$ max

7/ (a) quantum: E is discrete

classical: E is continous

(b) $$\frac{\Delta E}{E} = \frac{E_{n+1} - E_n}{E_n}$$. this $\to 0$ in the limit of high n.

In SHO case: $$\frac{(n+1+\frac{1}{2})\hbar\omega - (n+\frac{1}{2})\hbar\omega}{(n+\frac{1}{2})\hbar\omega} = \frac{\hbar\omega}{(n+\frac{1}{2})\hbar\omega}$$

$$\lim_{n\to\infty}\left(\frac{\hbar\omega}{(n+\frac{1}{2})\hbar\omega}\right) = 0.$$

in inf. well case: $$\frac{\frac{(n+1)^2\hbar^2\pi^2}{2mL^2} - \frac{n^2\ldots}{}}{\frac{n^2\hbar^2\pi^2}{2mL^2}} = \frac{(n+1)^2 - n^2}{n^2}$$

$$= \frac{\cancel{n^2} + 2n + 1 - \cancel{n^2}}{n^2}$$

$$\lim_{n\to\infty} \frac{2n+1}{n^2} = 0$$

So the "steps" between energy levels becomes small compared to the previous energy level — difficult (impossible?) to discern

* there's also a zero-point energy in QM. That is, E_{lowest} = bottom of well. This energy turns out to be ≈ 0 for most classical (large m, large L) systems since $\hbar \approx 10^{-34}$. $E_{gnd} = \frac{\hbar^2\pi^2}{2mL^2}$, $\frac{1}{2}\hbar\omega$

8/ must use the time-dependent Sch. eqn

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + U(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

since the wave fnc is $\Psi(x,t)$ and not $\psi(x)$.

(a) $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[Ae^{i(kx-\omega t)}\right] + 0 = i\hbar\frac{\partial}{\partial t}\left[Ae^{i(kx-\omega t)}\right]$$

$$-\frac{\hbar^2}{2m}A(ik)^2 e^{i(kx-\omega t)} = i\hbar A(-i\omega)\, e^{i(kx-\omega t)}$$

$$\boxed{\frac{\hbar^2 k^2}{2m} = \hbar\omega}$$

if this is true, $\Psi(x,t) = Ae^{i(kx-\omega t)}$ is a soln to Sch. eqn.

(b) $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[A\cos(kx-\omega t)\right] + 0 = i\hbar\frac{\partial}{\partial t}\left[A\cos(kx-\omega t)\right]$$

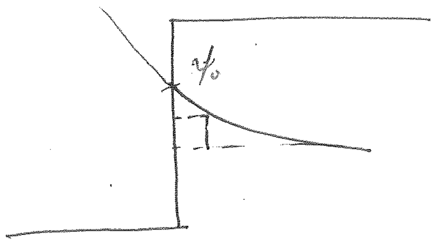
$$-\frac{\hbar^2}{2m}(-k^2)A\cos(kx-\omega t) = i\hbar(+\omega)A\sin(kx-\omega t)$$

$$\frac{\hbar^2 k^2}{2m}\cos(kx-\omega t) = i\hbar\omega\sin(kx-\omega t)$$

X

this doesn't work! to satisfy the Sch. eqn, Ψ must be a complex fnc.

9/ In the classically forbidden region, $\psi = \psi_0 e^{-\alpha x}$.



$\psi \to 0$ in limit that $x \to \infty$.

We define the penetration depth as the distance, $x = \delta$, at which ψ falls to e^{-1} of its value at the well edge,

$$\psi(x=\delta) = \psi_0 e^{-1}$$
$$= \psi_0 e^{-\alpha\delta}$$

therefore,

$$-1 = -\alpha\delta$$
$$\delta = \frac{1}{\alpha} \quad \text{where } \alpha = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$