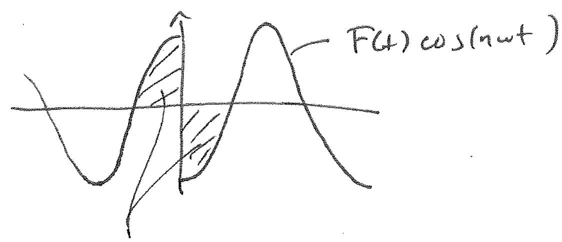
Exam 2 Solutions

1/ $$a_n = \frac{2}{T}\int F(t)\cos(n\omega t)\,dt = 0$$

- Here, $F(t)$ is an antisymmetric func. Therefore, we must build it from anti-symmetric funcs. Symmetric funcs <u>cannot</u> make anti-symmetric funcs.
- Also, consider the integrand: $F(t) \times \cos(n\omega t)$ = antisymm × symm = antisym. The integral of an anti-symmetric func over an even interval is zero. This is seen in the following graph:



these areas cancel

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin(n\omega t)\,dt$$

$$= \frac{\omega}{\pi}\left[\int_{-10}^{0} 3\sin(n\omega t)\,dt + \int_{0}^{10} -3\sin(n\omega t)\,dt\right]$$

$$= \frac{3\omega}{\pi}\left[-\frac{\cos(n\omega t)}{n\omega}\Big|_{-10}^{0} - \frac{-\cos(n\omega t)}{n\omega}\Big|_{0}^{10}\right]$$

$$= \frac{3}{\pi n}\left[-(1-(-1)^n) + ((-1)^n - 1)\right]$$

$$b_n = \frac{-12}{n\pi} \text{ if } n \text{ odd}, \quad 0 \text{ if } n \text{ is even}$$

2/ (a) $F(t) = \frac{1}{2}a_0 + \sum a_n \cos(n\omega t) + \sum b_n \sin(n\omega t)$

here, $a_n = 0$ and $b_n = -\frac{12}{n\pi}$ for n = odd, 0 otherwise

$$\boxed{\begin{array}{l} F(t) = \frac{-12}{\pi}\sin(\omega t) - \frac{4}{\pi}\sin(3\omega t) + \ldots \\ \text{here, } \omega = \pi/10 \\ F(t) = -\frac{12}{\pi}\sin\left(\frac{\pi t}{10}\right) - \frac{4}{\pi}\sin\left(\frac{3\pi t}{10}\right) + \ldots \end{array}}$$

(b) The steady state sln is $x = \sum \alpha_n x_{p_n}(t)$ where α_n corresponds to a_n, b_n.

$$x_{p_n}(t) = \frac{F_{0n}/m}{\left[(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2\right]^{1/2}} \cos(\omega_n t - \delta)$$

for $F(t) = F_{0_n}\cos(\omega_n t)$

here, $\omega_0 = \pi$, $F_{0n} = -\frac{12}{n\pi}$, $\omega_n = \frac{n\pi}{10}$, $m = 2$, $\beta = \frac{1}{2}$

$$x(t) = \frac{\frac{-12}{2\pi}\sin\left(\frac{\pi t}{10} - \delta_n\right)}{\left\{\left[\pi^2 - \left(\frac{\pi}{10}\right)^2\right]^2 + (4)\left(\frac{\pi}{10}\right)^2\left(\frac{1}{2}\right)^2\right\}^{1/2}} + \frac{\frac{-4}{2\pi}\sin\left(\frac{3\pi t}{10} - \delta_n\right)}{\left\{\ldots\left(\frac{3\pi}{10}\right)^2\ldots\left(\frac{3\pi}{10}\right)^2\ldots\right\}^2}$$

* note the standard solution is for $F(t) = F_0\cos(\omega t)$

Our driving force here is $F(t) = F_0\sin(\omega t)$

So. our slns are written in terms of sine, not cosines.

δ_n also has an ω dependence, not shown here

3/ $S = \int ds \qquad ds = \sqrt{dx^2 + dy^2}$

$$= \int \sqrt{dx^2 + dy^2}$$

$$= \int \underbrace{\sqrt{1 + y'^2}}_{f(y, y'; x)} \, dx \qquad \text{where } y' = \frac{dy}{dx}$$

to minimize this, use Euler eqn $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

$\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial y'} = \frac{1}{2}(1 + y'^2)^{-1/2}(2y')$

Since $\frac{\partial f}{\partial y} = 0 \qquad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\frac{\partial f}{\partial y'} = \frac{y'}{(1 + y'^2)^{1/2}} = \text{constant}, \; C$$

$$\frac{y'^2}{1 + y'^2} = C \qquad \text{solve for } y' \ldots$$

$$y'^2 = (1 + y'^2)\,C$$

$$y'^2 - Cy'^2 = C$$

$$y'^2 = \frac{C}{1 - C}$$

$$y' = C$$

$$\frac{dy}{dx} = C$$

$$\boxed{y = Cx + C_2}$$ a straight line

4/ The path a system takes is the one that minimizes the action, S. Here the action is defined as the integral

$$S = \int L\,dt \qquad \text{where } L = T - U$$

To minimize S, use the Euler-Lagrange eqn $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$

5/ (a) $L = T - U$ this is best done in cylindrical coordinates, ρ, z, ϕ

where $v^2 = \dot{\rho}^2 + \dot{z}^2 + \rho^2\dot{\phi}^2$

$T = \frac{1}{2}m(\dot{\rho}^2 + \dot{z}^2 + \rho^2\dot{\phi}^2)$ and $z = r^2$

$\dot{z} = 2r\dot{r}$ or in terms of ρ

$\dot{z} = 2\rho\dot{\rho}$

$T = \frac{1}{2}m(\dot{\rho}^2 + 4\rho^2\dot{\rho}^2 + \rho^2\dot{\phi}^2)$

$U = mgz = mg\rho^2$

$$\boxed{L = \frac{1}{2}m(\dot{\rho}^2 + 4\rho^2\dot{\rho}^2 + \rho^2\dot{\phi}^2) - mg\rho^2}$$

(b) 2 generalized coordinates : ρ & ϕ (or could be z & ϕ)

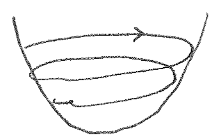
5 cont'd

Note: this can also be done in Cartesian, but it gets messy.

- $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

  need to incorporate $\left.\begin{array}{l} y^2 + x^2 = r \\ \text{and} \quad z = r^2 \end{array}\right\}$ relate $\dot{z}$ to $\dot{y}, \dot{x}$ to reduce parameters to 2

- We can't just say $L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$ to reduce the dimensionality. This confines it to x-z plane + won't predict a spiraling motion



which is a possible behavior. This also doesn't confine the particle to the bowl edges.

6/ $T = \frac{1}{2} M\dot{x}_1^2 + \frac{1}{2} M\dot{x}_2^2$

$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_2^2 + \frac{1}{2} k_2 (x_1 + x_2)^2$

$L = T - U$

$L = \frac{1}{2} M(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}\left[k_1 x_1^2 + k_3 x_2^2 + k_2 (x_1+x_2)^2\right]$

two genalized coordinates, x_1 & x_2

$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}$:

$$\frac{\partial L}{\partial x_1} = -\frac{1}{2} k_1 x_1 2 - \frac{1}{2} k_2 2(x_1 + x_2)$$

$$= -k_1 x_1 - k_2 (x_1 + x_2)$$

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{2} M (2\dot{x}_1) = M\dot{x}_1$$

$$\frac{d}{dt}(\ ) = M\ddot{x}_1$$

$$-k_1 x_1 - k_2 (x_2 + x_1) - M\ddot{x}_1 = 0$$

$$\boxed{\longrightarrow \ddot{x}_1 + \frac{x_1}{M}(k_1 + k_2) + k_1 x_2 = 0}$$

$\frac{\partial L}{\partial x_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2}$:

$$\frac{\partial L}{\partial x_2} = -\frac{1}{2} k_3 (2x_2) - \frac{1}{2} k_2 (2)(x_1 + x_2)$$

$$\frac{\partial L}{\partial \dot{x}_2} = \frac{1}{2} M (2\dot{x}_2)$$

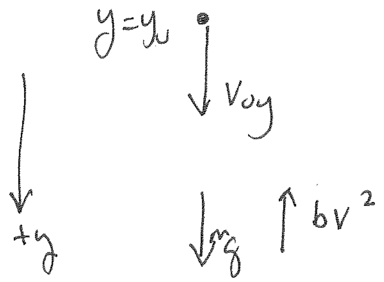
$$\frac{d}{dt}(\ ) = M\ddot{x}_2$$

$$-k_3 x_2 - k_2 (x_1 + x_2) - M\ddot{x}_2 = 0$$

$$\boxed{\ddot{x}_2 + \frac{x_2}{M}(k_2 + k_3) + k_2 x_1 = 0}$$

Exam 1 revisited

1/



$$F_y = m\ddot{y}$$

$$+mg - bv_y^2 = m\ddot{y} \qquad v_y = \dot{y}$$

$$+mg - bv^2 = m\dot{v} \qquad \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt}$$

$$+mg - bv^2 = m\frac{dv}{dy}v \qquad = \frac{dv}{dy}v$$

$$dy = \frac{mv\,dv}{mg - bv^2}$$

$$\int_{y_0}^{y} dy = \int_{v_0}^{v} \frac{v\,dv}{g - \frac{b}{m}v^2}$$

$$y - y_0 = \int_{v_0}^{v} \frac{v\,dv}{g - \frac{b}{m}v^2}$$

2/ for SHO $U = \frac{1}{2}kx^2$

here $U = U_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - U_0$

if $\frac{(x-x_0)}{\delta} << 1$ then $e^{-\frac{(x-x_0)}{\delta}} \approx 1 + \left[\frac{-(x-x_0)}{\delta}\right] + \frac{1}{2}\left[\frac{-(x-x_0)}{\delta}\right]^2 + \dots$

$$U \approx U_0\left[1 - \left(1 + \frac{-(x-x_0)}{\delta} + \frac{1}{2}\frac{(x-x_0)^2}{\delta^2} + \dots\right)\right]^2 - U_0$$

$$\approx U_0\left[\frac{(x-x_0)}{\delta}-\frac{(x-x_0)^2}{\delta^2}+\dots\right]^2-U_0$$

$$\approx U_0\frac{(x-x_0)^2}{\delta^2}+\dots\underbrace{\left[\text{higher order terms of } \frac{(x-x_0)^2}{\delta^2}\right]}_{\text{these are negligible}}-U_0$$

$$\boxed{\begin{aligned} U &\approx \frac{U_0}{\delta^2}(x-x_0)^2 \\ U &\approx \frac{1}{2}kx^2 \end{aligned}}$$

where $\frac{1}{2}k=\frac{U_0}{\delta^2}$

$x^2=(x-x_0)^2$

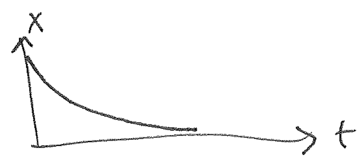
oscillation $\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{2U_0}{\delta^2 m}}$

3/

(a) critically damped, undriven oscillator

$\omega_0=\sqrt{13}$

$\beta=\sqrt{13}$



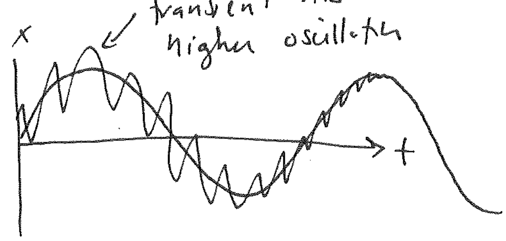
(b) $2\beta=2\sqrt{13}\rightarrow\beta=\sqrt{13}$; $\omega_0^2=36\rightarrow\omega_0=6$ $\Big\}$ $\beta<\omega_0$

$2t=\omega t$

$\omega=2$

$\underbrace{\text{under damped}}_{\omega_1=\sqrt{\omega_0^2-\beta^2}}$, $\underbrace{\text{driven}}_{\omega=2}$ oscillator

$\omega_1\approx 5$



(c) un-damped, driven oscillator

$\omega_0=\sqrt{5}$ $\omega=10$