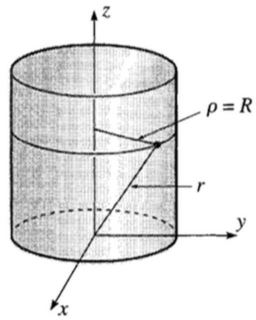
two significant corrections: $\vec{r}_1$ and $\dot{\vec{r}}_1$ in the double pendulum, and $\frac{1}{2}\Delta T$ in the periodic pulse

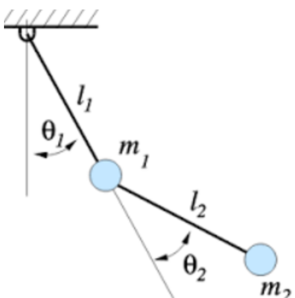
Particle confined to move on a cylinder.[1] A particle is constrained to move on the surface of a frictionles cylinder of radius $\rho = R$. There is also a force $\vec{F}_1 = k\vec{r}$ directed towards the origin. $\vec{F}_1$'s associated potential energy is $U = \frac{1}{2}kr^2$ where $r = \sqrt{R^2 + z^2}$. Determine the motion of this particle.



(a) Write the Lagrangian. Use cylindrical coordinates.
(b) How many generalized coordinates are there?
(c) Determine Lagrange's equation(s) of motion.
(d) Interpret the(se) equation(s). That is, *describe* using text what the motion is like.

Double pendulum. The double pendulum looks like a simple system. However, its behavior is chaotic: its motion is sensitive to initial conditions and "although a system obeys deterministic equations of motion (such as Newton's laws) detailed future behavior may, as a practical matter, be unpredictable."[2]

Consider the following double pendulum. It has masses m_1 and m_2. The two arms are equally long $\ell_1 = \ell_2 = \ell$.



The position, velocity, and speed squared of m_1 are

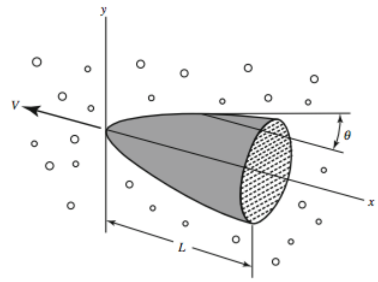
$$\begin{aligned} \vec{r}_1 &= \ell\cos\theta_1\hat{x} + \ell\sin\theta_1\hat{y} \\ \dot{\vec{r}}_1 &= -\ell\dot{\theta}_1\sin\theta_1\hat{x} + \ell\dot{\theta}_1\cos\theta_1\hat{y} \\ r_1^2 &= \dot{\vec{r}}_1\cdot\dot{\vec{r}}_1 = \ell^2\dot{\theta}_1^2 \end{aligned}$$

where $+\hat{x}$ down and $+\hat{y}$ is to the right.

(a) Determine the position, velocity and speed squared of m_2.
(b) Write the Lagrangian for this system.
(c) How many generalized coordinates are there and what are they?
(d) Determine one of the equations of motion. (Note that the equations are coupled.)

[1]from Taylor, Classical Mechanics (University Science Books).
[2]Taylor, Classical Mechanics, (University Science Books, Mill Valley CA, 2005) p457

A common aerodynamic problem is to find a surface that minimizes drag force. This surface is described by a curve, $y(x)$, rotated around the x-axis.

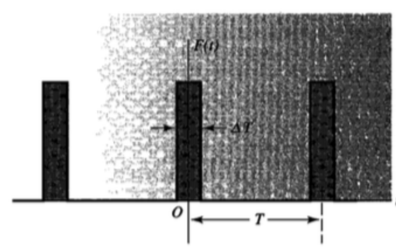


If the nosecone is long and thin ($\frac{dy}{dx} \ll 1$), the drag force is

$$F_{\text{drag}} \approx 4\pi\rho V^2 \int_0^L \left(\frac{dy}{dx}\right)^3 y dx$$

Determine $y(x)$, or the d.e. for $y(x)$, that minimizes the drag.[3]

Periodic pulse.[4] A harmonic oscillator is driven by a succession of rectangular pulses.



$$\begin{aligned} F(t) &= F_0 \quad \text{for } -\frac{1}{2}\Delta T \le t \le +\frac{1}{2}\Delta T \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Write the first three terms of the Fourier series.

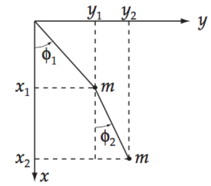
Note: $a_0 = \frac{2}{T}\int_{-T/2}^{T/2} F(t)dt$. a_n and b_n are as previously defined.

The solution is

$$F_{ext}(t) = F_0\left[\frac{\Delta T}{T} + \frac{2}{\pi}\sin\left(\pi\frac{\Delta T}{T}\right)\cos(\omega t) + \frac{2}{2\pi}\sin\left(2\pi\frac{\Delta T}{T}\right)\cos(2\omega t) + \frac{2}{3\pi}\sin\left(3\pi\frac{\Delta T}{T}\right)\cos(3\omega t) + \cdots\right]$$

[3]this version from Dym, Shames, *Solid Mechanics* (Springer 2013)
[4]this version from Fowles and Cassidy, Analytical Mechanics

Solution to double pendulum



If we take (ϕ_1, ϕ_2) as our generalized coordinates, the x,y coordinates of the two masses are

$$\left.\begin{aligned} x_1 &= \ell \cos\phi_1 \\ y_1 &= \ell \sin\phi_1 \end{aligned}\right] \tag{1}$$

$$\left.\begin{aligned} x_2 &= \ell \cos\phi_1 + \ell\cos\phi_2 \\ y_2 &= \ell \sin\phi_1 + \ell\sin\phi_2 \end{aligned}\right] \tag{2}$$

Using (1) and (2), we find the kinetic energy of the system to be

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) \\ &= \frac{m}{2}\ell^2\left[\dot{\phi}_1^2 + \dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\left(\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\right)\right] \\ &= \frac{m}{2}\ell^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right)\right] \end{aligned} \tag{3}$$

Therefore, the Lagrangian is

$$L = m\ell^2\left[\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 + \dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right)\right] + mg\ell\left[2\cos\phi_1 + \cos\phi_2\right] \tag{5}$$

from which

$$\left.\begin{aligned} \frac{\partial L}{\partial \phi_1} &= m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - 2mg\ell\sin\phi_1 \\ \frac{\partial L}{\partial \dot{\phi}_1} &= 2m\ell^2\dot{\phi}_1 + m\ell^2\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right) \\ \frac{\partial L}{\partial \phi_2} &= -m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - mg\ell\sin\phi_2 \\ \frac{\partial L}{\partial \dot{\phi}_2} &= m\ell^2\dot{\phi}_2 + m\ell^2\dot{\phi}_1\cos\left(\phi_1 - \phi_2\right) \end{aligned}\right] \tag{6}$$

The Lagrange equations for ϕ_1 and ϕ_2 are

$$\boxed{2\phi_1 + \ddot{\phi}_2\cos\left(\phi_1 - \phi_2\right) + \dot{\phi}_2^2\sin\left(\phi_1 - \phi_2\right) + 2\frac{g}{\ell}\sin\phi_1 = 0} \tag{7}$$

$$\boxed{\ddot{\phi}_2 + \ddot{\phi}_1\cos\left(\phi_1 - \phi_2\right) - \dot{\phi}_1^2\sin\left(\phi_1 - \phi_2\right) + \frac{g}{\ell}\sin\phi_2 = 0} \tag{8}$$