Analytical Mechanics - Extra Problems

Physics 105, F17

(R) are review problems. Review problems are those that have already been covered in prior courses, mostly Intro to Physics I and II. Some are math problems that you've run into in various math classes. Refer to your intro physics and math texts when necessary.

Unmarked problems are Analytical Mechanics problems from other texts or a variation on those from our text.

1. (R) A particle moves such that its position is

$$\vec{r} = R\cos(\omega t)\hat{x} + R\sin(\omega t)\hat{y}$$

(a) Describe the motion of this particle. Provide a sketch.
(b) Determine an expression for the particle's velocity.
(c) Determine an expression for the particle's acceleration.
(d) What's the magnitude of the acceleration? Compare the direction of the acceleration to its position (justify your answer using the expressions for $\vec{r}$ and $\vec{a}$).

2. (R) This is a review of the equations for a projectile motion. In this special case of 2D motion, the force is given by

$$\vec{F} = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$$

At $t = 0$, a particle is at $\vec{r} = x_0\hat{x} + y_0\hat{y}$ and has an initial velocity $\vec{v}_0 = v_0\cos\theta\hat{x} + v_0\sin\theta\hat{y}$.

(a) Derive the following expression for the particle's motion in the horizontal (x) direction. Start with Newton's 2nd law.

$$x(t) = x_0 + (v_0\cos\theta)t \tag{1}$$

(b) Repeat for the vertical (y) motion, where

$$y(t) = y_0 + (v_0\sin\theta)t - \frac{1}{2}gt^2 \tag{2}$$

(c) Use the above equations to determine an expression for the trajectory, $y(x)$.

(d) The range, R, is the horizontal distance traveled by the projectile during its flight. Assuming that the projectile lands at the same vertical position ($y_f = y_0$) derive the following:

$$R = \frac{v_0^2}{g}\sin(2\theta)$$

(e) The time of flight, T, is the total time the projectile is in flight. Assuming that the projectile lands at the same vertical position, derive the following:

$$T = \frac{2v_0}{g}\sin\theta$$

3. (R) What is conservation of momentum?

Write 2-4 sentences that describe this principle. Include an equation.

If you're having trouble with describing the principle in general, give an specific example that involves conservation of momentum. Include equations and values in your example.

4. The terminal speed of a 70kg skydiver in spread eagle position is around 50 m/s (about 120mph). Assume quadratic air resistance.

   Find the skydiver's speed at times $t = 1, 5, 10, 20, 30$ seconds after he jumps from a stationary balloon. Compare with the corresponding speeds if there were no air resistance.

   (Taylor; this is in your notes, not your text)

5. (R) A volcanic ash flow is moving across a horizontal ground when it encounters a 10^o upslope. It's observed to travel 920m on the upslope before coming to rest. The volcanic ash contains trapped gas, so the force of friction with the ground is very small and can be ignored. At what speed was the ash flow moving just before encountering the upslope?

6. (R) (a) Write the equation for the universal law of gravitation

   (b) Write the equation for the gravitational potential energy (the general case, not mgh).

   (c) Show that $F = -dU/dr$ for gravity.

7. (a) What is energy conservation? Write 2-4 sentences to describe this principle.

   (b) What is a conservative force? Write 2-4 sentences to describe this principle.

   If you?re having trouble describing the general principles, give specific examples.You may also cite other people's words, but include your own also.

8. (a) What is a Taylor series? Write 1-4 sentences and an equation.

   (b) Provide an example of a function expressed as a Taylor series. Be specific about your function and the point about which the series is taken. (For example, $f(x) = \ln(1+x)$ about the point $x = 0$.)

9. (R) The position of a particle is given by

   $$x = 4\cos(3\pi t + \pi)$$

   where x is in meters and t is in seconds.

   Determine the (a) frequency, (b) period, (c) amplitude, and (d) phase constant of this motion. Determine the (e) phase of the motion and (f) position of the particle at $t = 2.5$s. (Serway and Jewett, modified)

10. (R) The mass of deuterium molecule (D_2 is twice that of a hydrogen molecule (H_2). If the vibrational frequency of H_2 is 1.3×10^{14}Hz, what is the vibrational frequency of D_2? Assume the "spring constant" of attracting forces is the same for the two molecules (Serway and Jewett)

11. (R) A 50g object connected to a spring with a force constant of 35 N/m oscillates with an amplitude of 4cm on a frictionless, horizontal surface.

    Find (a) the total energy of the system and (b) the speed of the object when its position is 1cm. Find (c) the kinetic energy and (d) potential energy when its position is 3cm. (Serway and Jewett)

12. A pendulum is subject to a restoring force of $F = -mg\sin\theta$ and a resistive force of $F = 2m(\sqrt{g/\ell})(\ell\dot{\theta})$.

    (a) Write the differential equation for θ. That is, $\ddot{\theta} = \ldots$

    (b) Is this pendulum underdamped, critically damped, or overdamped? From your answer, write the solution for $\theta(t)$. (No need to *solve* for it, just write it down.)

    (c) At $t = 0$, the pendulum is pulled back to $\theta = \alpha$ and released from rest, $\dot{\theta} = 0$. Use these initial conditions to determine the constants (A, A_1, A_2, or B) in $\theta(t)$.

    (d) Calculate $\dot{\theta}(t)$.

    **note: this was corrected at 10am March 16.

    (Hint: this is an example in your text.)

13. A spring suspension system has β equal to one-tenth its critical value. Its undamped frequency is ω_0 and it's driven at $\omega = \omega_0/2$. Determine the

(a) damped frequency
(b) resonant frequency
(c) quality factor
(d) steady state frequency
(e) the amplitude at steady-state.

14. The amplitude of a driven damped oscillator is given by

$$D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}$$

Let $A = 1$ and $\omega_0 = 100$. It can be driven at any frequency, but you're interested in the range of $1 < \omega < 200$.

(a) Plot $D(\omega)$ for $\beta = 1, 2, 5$, and 10. Plot these 4 curves on one graph. Use a range of $0 < \omega < 200$ and steps of about 0.2. You're welcome to use Matlab, profit, or Excel.

(b) Fill in the following table with values. You're welcome to calculate these values from the equations or by zooming in on your graph. (Play close attention to $\beta = 10$)

β	amplitude	ω that leads to largest amplitude	$\Delta\omega$	Q
1				
2				
5				
10				

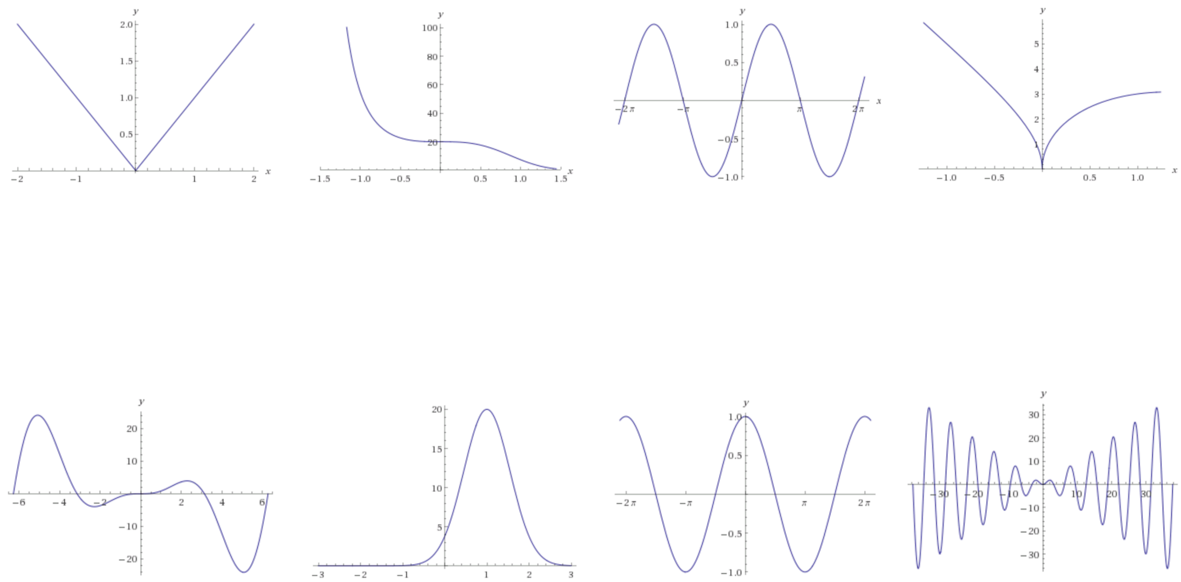
(c) Draw lines indicating $\Delta\omega$ on your curves.

15. A symmetric function is one where $g(-x) = g(x)$. Anti-symmetric functions have $g(-x) = -g(x)$. We sometimes use the terms *even* and *odd* for symmetric and anti-symmetric. For each of the following (a)-(p), determine if

    i. it's symmetric, anti-symmetric, or neither.
    ii. an integral taken from $x = \frac{-\pi}{2} \to \frac{+\pi}{2}$ is zero or non-zero.

    Calculations or explanations aren't expected.

    (a)-(h)



    (i) x^3 (j) $\sin x$ (k) $\cos x$ (l) $x + x^2$
    (m) $x \sin x$ (n) $x \cos x$ (o) e^{-x^2} (p) $e^{-(x-2)^2}$

16. (a) What is Snell's law? Your answer should include 1-2 sentences, one equation and one diagram.

    (b) What is Fermat's principle? Your answer should include 1-2 sentences, one diagram, and possibly one equation.

17. (R) (a) Give the general equation for integration by parts.

    (b) Give a specific example of integration by parts. That is, give an actual function for u and v.

    You're welcome to copy this straight out of your math text or some internet source. Give credit where it's due.

18. Write the equation(s) for a helix with a fixed radius.

19. Find the function $y(x)$ that minimizes the integral

$$\int_1^2 \frac{y'^2 dx}{x^3}$$

    the end points are (1,0) and (2,15).

    (The answer is $y = x^4 - 1$)

20. In Cartesian coordinates, the speed of an object is given by

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

    In cylindrical coordinates, $x = \rho \cos\phi$, $y = \rho \sin\phi$ and $z = z$.

    (a) Determine the expression for v^2 in cylindrical coordinates. That is, using ρ, ϕ, z.

(b) What is v^2 if the motion is confined to the surface of cylinder, $\rho = \rho_0$?

(c) What is v^2 if the object doesn't circle the z-axis?

21. Consider two points on a right angle cone, where $z = \rho$. What is the shortest path between the points? Use cylindrical coordinates and determine the equation for $\phi(\rho)$.

    Hint: the final integral that you'll get to is

$$\int d\phi = \int \frac{C d\rho}{\rho\sqrt{\rho^2 - C^2}}$$

    Solve this integral with the substitutions
    $\cos\theta = C/\rho$,
    $d\rho = C(\tan\theta/\cos\theta)d\theta$,
    and $\rho^2 - C^2 = C^2\tan^2\theta$.
    You should come up with $\rho\cos(\phi) = C$.

    C will turn out to be ρ_0, the initial radius. As noted by your text, this is the shortest path on a volcano.

22. Use Euler's approximation to find a numerical solution to

$$\dot{x} = 1 - 1.5t^2$$

    Determine values of x for $t = 0$ to 2s. Let $x(t = 0) = 0$. Start with 10 time steps. Try at least 2 other smaller time intervals, until your answer converges. Compare to the exact solution.

    Submit:
    (1) your code. It should work; I'll run it.
    (2) A graph of your numerical solutions for $x(t)$ with the different time steps. Add the exact solution.

23. Use Euler's approximation to solve a second order D.E. Simply do Euler's method twice. Solve

$$\ddot{y} = -g$$

    Rewrite $\ddot{y} = -g$ as two first order d.e.'s, $\dot{v} = -g$ and $\dot{y} = v$. Start with $y(0) = 5$ $\dot{y}(0) = 20$ and look at $t = 0 \to 5$. Start with 10 time steps, and try smaller intervals until your answer converges. Compare to the exact solution.

    Submit:
    (1) your code. It should work; I'll run it.
    (2) A graph of your numerical solutions for $y(t)$ with the different time steps. Add the exact solution.

24. Start your project. Submit

    (1) a list of names of the people in your group.
    (2) a drawing of the system you want to work on. Add labels
    (3) the Lagrangian for your system and the equations of motion for your system. Or the equation of motion via Newton's second law.
    (4) for one of your equations of motion, calculate values for the first 3 positions. Use a time step of 0.2.

    One submission per group.

25. Submit solutions to exam 2.

26. What is *Noether's theorem* regarding symmetry and conservation laws? Look it up some place reliable on the web. Write 1-4 sentences.

    You're welcome to copy it straight off a site. If you do so, use quotes and give credit where it's due (give a reference). Make sure your site is a reliable one (something from an .edu, .gov, wikipedia isn't bad if it has real references)

27. Skim section 7.9 and 7.10. Give definitions or identify/write equations for

    (a) *generalized coordinates* and *generalized momenta*.

    (b) the Hamiltonian of a system.

    (c) Hamilton's equations of motion.