Lagrangian and Hamiltonian dynamics

7.4 modified. A particle moves under the influence of a force $f = A\rho^{\alpha-1}$ directed towards the origin and $A, \alpha > 0$. Choose ρ and θ, that is cylindrical coordinates, as generalized coordinates. Determine

(a) the potential energy associated with this force
(b) the generalized momenta and generalized forces for this system
(c) whether momenta are conserved
(d) whether total energy is conserved.

Mass sliding on a cone.[1] A mass is constrained to move on the frictionless surface of a vertical cone where $\rho = cz$. There is a gravitational field vertically down.

(a) Write the Lagrangian for this system. Use cylindrical coordinates and incorporate the relationship between z and ρ.
(b) Determine the generalized momenta.
(c) Determine the Hamiltonian.
(d) Write Hamilton's equations of motion.

(e) Show that the particle will stay at a fixed height and move in a horizontal circle if it is started with $p_z = 0$ and $p_\phi = \pm\sqrt{m^2c^2gz^3}$.

7.28 A particle of mass m is attracted to a force center with a force of magnitude k/r^2. Use polar (i.e. cylindrical) coordinates and find Hamilton's equations of motion.

Small oscillations and simple harmonic motion.

(a) A bottle floats and bobs in a lake.

At equilbrium, the bottle is submerged at a depth of d_0 such that $mg = \rho g A d_0$. When the bottle is displaced from equilibrium, its equation of motion, according to Newton's second law, is

$$m\ddot{x} = mg - \rho g A(d_0 + x).$$

Show that this equation of motion reduces to

$$\ddot{x} + \omega_0^2 x = 0.$$

Determine the period of the oscillation.

(b) A cube balanced on a cylinder has a potential energy

$$U(\theta) = mg[(r+b)\cos\theta + r\theta\sin\theta]$$

Show that for small angles, this leads to simple harmonic motion. Determine the period of these oscillations.

[1] Taylor, Classical Mechanics, p533