Generalized coordinates

You can describe the motion using a set of *generalized coordinates*, q_i. These coordinates can be Cartesian, spherical, polar, or you can make up your own! Whatever makes sense.

You need $s = nD - m$ coordinates. Here n is the number of particles, D is the dimension, and m is the number of constraints. The number s is called the *degrees of freedom* of the system.

1. How many generalized coordinates do you need for one *free* particle in 2D?
2. How many generalized coordinates do you need for one free particle in 3D?
3. How many generalized coordinates do you need for a pendulum moving in 3D?
4. Two particles have a fixed separation distance (like a barbell). Except for their connection, they are free to move in 3D. You want to describe the motion of particle 1 and particle 2. How many generalized coordinates do you need?

Lagrange's equations of motion

For each of the following write the specific expression for the Lagrangian, and determine the equation(s) of motion (that is, write the d.e.s that determine $q_i(t)$).

1. A train starts from rest with a constant acceleration of a. A pendulum is hung from one of car's ceiling. The position of the pendulum bob in Cartesian coordinates is

   $x = \frac{1}{2}at^2 + \ell \sin\theta$
   $y = -\ell \cos\theta$

   That is, it's the position of the pivot point plus the deflection of the pendulum.

   (a) Write the Lagrangian. (Note that T requires taking time derivatives of the position.)

   (b) Determine the equations of motion.

   (c) What is the equilibrium position of this pendulum?

   (e) Does this make sense? That is, can you take a limit that makes this look like a problem you already know the solution to, and does it behave correctly in that limit?

2. A bead can slide without friction along a circular hoop of radius R. The hoop spins with a fixed speed around its vertical axis.

   Given the situation, it's best to use spherical coordinates. In spherical coordinates, $v^2 = \dot{r}^2 + r^2\dot{\phi}^2 + r^2\dot{\theta}^2 \sin^2\phi$.

   (a) Write the Lagrangian. Let the center of the circle be $y = 0$ for the gravitational potential energy.

   (b) Determine the equation of motion.

   (c) If this is a 3D problem, why is do we only have 1 equation? What are the two constraints involved?

   (d) What are the equilibium positions of this pendulum?

more Lagrangian problems
analyzing solutions, coupled systems

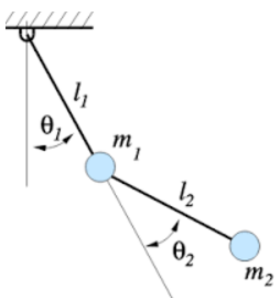
1. The two equations of motion for the sliding block on a frictionless wedge problem

$$\ddot{q}_1 = \frac{g\sin\alpha}{1-\frac{m\cos^2\alpha}{M+m}}$$
$$\ddot{q}_2 = \frac{-mg\cos\alpha\sin\alpha}{M+m(1+\cos\alpha)}$$

What occurs in the limit that
(a) $\alpha \to 0$?
(b) $\alpha \to \pi/2$?
(a) $m \ll M$?
(d) $m \gg M$?

2. The double pendulum looks like a simple system. However, its behavior is chaotic: its motion is sensitive to initial conditions and "although a system obeys deterministic equations of motion (such as Newton's laws) detailed future behavior may, as a practical matter, be unpredictable."[1]

Consider the following double pendulum. It has masses m_1 and m_2. Let two arms be of equal length $\ell_1 = \ell_2 = \ell$.



The position, velocity, and speed squared of m_1 are

$$\vec{r}_1 = \ell\cos\theta_1\hat{x} - \ell\sin\theta_1\hat{y}$$
$$\dot{\vec{r}}_1 = -\ell\dot{\theta}_1\sin\theta_1\hat{x} - \ell\dot{\theta}_1\cos\theta_1\hat{y}$$
$$r_1^2 = \dot{\vec{r}}_1\cdot\dot{\vec{r}}_1 = \ell^2\dot{\theta}_1^2$$

(a) Determine the position, velocity and speed squared of m_2.
(b) Write the Lagrangian for this system.
(c) How many generalized coordinates are there and what are they?
(d) Determine one of the equations of motion. (Note that the equations are coupled.)

[1] Taylor, Classical Mechanics, (University Science Books, Mill Valley CA, 2005) p457

3. The following is an example of a set of coupled differential equations that can be uncoupled:

$$\ddot{y} = \omega\dot{z}$$
$$\ddot{z} = \omega(\frac{E}{B} - \dot{y})$$

(a) Uncouple them to write two d.e.'s: one just in y and one just in z.

Start by taking the time derivative of the one of equations and substitute into the other. The repeat the other way.

(b) The above d.e.'s are third order d.e.'s. Re-write them using $v_y = \dot{y}$ and $v_z = \dot{z}$ to come up with second order d.e.s