due Friday Mar 23 2018, beginning of class

Complex numbers: a review

Brief references:
Appendix K of *Modern Physics, 2nd ed* Harris
Ch 2.2 Complex Notation of *Basic electronics, 2nd ed*, Meyer

A complex number, z, has real and imaginary parts. A common representation a complex number is $z = a + ib$. Here, a is the real part and b is the imaginary part, otherwise referred to as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. Operations often use the complex conjugate, z^*. The conjugate replaces any i with $-i$.

1. Identify the real and imaginary parts of the following:
   (a) $z_a = 1 - 2i$
   (b) $z_b = 4i - 3$
   (c) $z_c = 5\cos\alpha + 5i\sin\alpha$
   (d) z_a^* (use z_a above)
   (e) $z_d = \frac{1}{6+3i}$ *(Note: The answer isn't 6 and 3. You'll need to re-write this in the form $z = a + ib$. A fool-proof way to do this is to multiply it with $\frac{z^*}{z^*}$.)*

Simple addition and subtraction of complex numbers follows the same rules as vector operations. That is, treat the Re part and the Im part as two orthogonal components.

2. Let $z = a + ib$, $z_e = 2 + 4i$, and $z_f = 3 - i$
   (a) Calculate $z_e + z_f$
   (b) Calculate $-3z_f + iz_e$
   (c) Show that $z - z^* = 2Im(z)$
   (d) Show that $z + z^* = 2Re(z)$

We can graph z in the complex plane, where the real part on the horizontal axis and the imaginary part is on the vertical axis. This also means that z's magnitude and direction can be calculated in a manner similar to the magnitude of a vector, $|z| = \sqrt{a^2 + b^2}$ and $\theta = \arctan(b/a)$.

3. For each of the following, plot z on the complex plane. Calculate $|z|$ and its angle ϕ CCW from the +Re axis. Give ϕ in terms of multiples of π. For example, $\pi/4$.
   (a) $1 + i$
   (b) i
   (c) $-2 + i$
   (d) -1
   (e) $-1 - 2i$
   (f) $+3 - i$
   (g) $6\cos(\frac{3\pi}{4}) + 6i\sin(\frac{3\pi}{4})$
4. A more general way of finding $|z|$ is $|z| = \sqrt{z^* z}$. Let $z = a + ib$. Show that $\sqrt{z^* z} = \sqrt{a^2 + b^2}$.
5. For each, plot z on the complex plane. Calculate $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$
   (a) $|z| = 3, \phi = \pi/6$
   (b) $|z| = 2, \phi = -\pi/3$
   (c) $|z| = 1, \phi = 3\pi/2$
   (d) $|z| = R$. Let ϕ be somewhere in the first quadrant.

From the above, it can be seen that the complex number is also $z = |z|(\cos\phi + i\sin\phi)$. If we use Euler's equation, $e^{i\phi} = \cos(\phi) + i\sin(\phi)$, we can find z in polar form, $z = |z|e^{i\phi}$.

5. Show that $e^{i\phi} = \cos(\phi) + i\sin(\phi)$. Start with the Taylor series for e^x, $\cos(x)$, and $\sin(x)$ where $x = i\phi$.

6. Show that $\cos(\phi) = \frac{1}{2}(e^{i\phi} + e^{-i\phi})$. Keep in mind that $e^{-i\phi} = \cos(\phi) - i\sin(\phi)$.

7. Show that $\sin(\phi) = \frac{1}{2i}(e^{i\phi} - e^{-i\phi})$.

8. Revisit (3). Write each in polar form.

9. (a) Let $z_g(t) = 3e^{i\omega t}$, where $\omega = \pi/12$. Plot $z(0), z(2), z(4)$, and $z(6)$ on one graph. Describe what happens to z as a function of time.

   (b) Let $z_h(t) = 3e^{-i\omega t}$, where $\omega = \pi/12$. Plot $z(0), z(2), z(4)$, and $z(6)$ on one graph. Describe what happens to z as a function of time.