Phys 125
Boyda

Solutions to problems from Chapters 10 and 12.

1. The ground state wavefunction for hydrogen is

$$\psi(\vec{r}) = R_{10}(r)Y_{00}(\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0},$$

where $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ and a_0 is the Bohr radius. The state is spherically symmetric – there is no dependence on θ or ϕ.

$$\langle x \rangle = \int d^3\vec{r}\, \psi^*(\vec{r})\, x\, \psi(\vec{r}) = \int d^3\vec{r}\, x \frac{1}{\pi a_0^3} e^{-2r/a_0} = 0,$$

because this is an odd function with respect to $x \to -x$ and we are integrating over the whole x axis (and indeed, the whole of three-space).

$$\begin{aligned}
\langle x^2 \rangle &= \int d^3\vec{r}\, \psi^*(\vec{r})\, x^2\, \psi(\vec{r}) \\
&= \int r^2 dr \sin\theta d\theta d\phi\, (r\sin\theta\cos\phi)^2 \frac{1}{\pi a_0^3} e^{-2r/a_0} \\
&= \frac{1}{\pi a_0^3} \int_0^\infty dr\ r^4 e^{-2r/a_0} \int_0^\pi d\theta\ \sin^3\theta \int_0^{2\pi} d\phi\ \cos^2\phi \\
&= \frac{1}{\pi a_0^3} \left(\frac{3a_0^5}{4}\right)\left(\frac{4}{3}\right)(\pi) = a_0^2.
\end{aligned}$$

Therefore,

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = a_0.$$

To compute $\langle p_x \rangle$ and $\langle p_x^2 \rangle$ we will need x derivatives of ψ. First note that since $r = \sqrt{x^2+y^2+z^2}$,

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2}(2x)}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}.$$

Using this repeatedly,

$$\begin{aligned}
\frac{\partial}{\partial x}\psi(\vec{r}) &= \frac{\partial}{\partial x}\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0} \\
&= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left(-\frac{1}{a_0}\right)\frac{\partial r}{\partial x} \\
&= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left(-\frac{x}{ra_0}\right); \\
\frac{\partial^2}{\partial x^2}\psi(\vec{r}) &= \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left(-\frac{x}{ra_0}\right)\right] \\
&= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left(-\frac{x}{ra_0}\right)^2 + \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\frac{\partial}{\partial x}\left(-\frac{x}{ra_0}\right) \\
&= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left[\left(-\frac{x}{ra_0}\right)^2 - \frac{1}{a_0}\left(\frac{r - x\frac{x}{r}}{r^2}\right)\right] \\
&= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left[\left(\frac{x^2}{r^2 a_0^2}\right) - \frac{1}{a_0}\left(\frac{r^2 - x^2}{r^3}\right)\right] \\
&= \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left[\left(\frac{x^2}{r^2 a_0^2}\right) - \left(\frac{y^2 + z^2}{a_0 r^3}\right)\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle p_x \rangle &= \int d^3\vec{r}\,\psi^*(\vec{r})\,(-i\hbar)\frac{\partial}{\partial x}\,\psi(\vec{r}) \\
&= -i\hbar\int d^3\vec{r}\,\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left(-\frac{x}{ra_0}\right) = 0,
\end{aligned}$$

for the same reason $\langle x \rangle = 0$.

$$\begin{aligned}
\langle p_x^2 \rangle &= \int d^3\vec{r}\,\psi^*(\vec{r})\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi(\vec{r}) \\
&= -\hbar^2\int d^3\vec{r}\,\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left[\left(\frac{x^2}{r^2 a_0^2}\right) - \left(\frac{y^2 + z^2}{a_0 r^3}\right)\right] \\
&= -\hbar^2\int r^2 dr\sin\theta d\theta d\phi\,\frac{1}{\pi a_0^3}e^{-2r/a_0}\left[\left(\frac{(r\sin\theta\cos\phi)^2}{r^2 a_0^2}\right) - \left(\frac{(r\sin\theta\sin\phi)^2 + (r\cos\theta)^2}{a_0 r^3}\right)\right] \\
&= \frac{\hbar^2}{3a_0^2}.
\end{aligned}$$

Finally,

$$\Delta p_x = \frac{\hbar}{a_0\sqrt{3}}.$$

Note that $\Delta x \Delta p_x = \hbar/\sqrt{3} > \hbar/2$, so the uncertainty principle is satisfied. By spherically symmetry, we would find the same results for the y and z axes (or any other rectilinear coordinate axes we choose).

The ground state is an angular momentum eigenstate with eigenvalue zero, so the electron is not tracing any well-defined classical orbit. But it is moving with some non-zero (probability distribution of) linear momenta around the proton.

10.2 For the hydrogen atom in $|n, l, m\rangle$ basis, we are given

$$|\psi(t=0)\rangle = \frac{4}{5}|1,0,0\rangle + \frac{3i}{5}|2,1,1\rangle.$$

Applying the time evolution operator,

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \frac{4}{5}e^{-iE_1 t/\hbar}|1,0,0\rangle + \frac{3i}{5}e^{-iE_2 t/\hbar}|2,1,1\rangle.$$

$$\begin{aligned}
\langle E\rangle &= \left|\frac{4}{5}e^{-iE_1 t/\hbar}\right|^2 E_1 + \left|\frac{3i}{5}e^{-iE_2 t/\hbar}\right|^2 E_2 \\
&= \frac{16}{25}\left(-\frac{\mu c^2\alpha^2}{2}\right) + \frac{9}{25}\left(-\frac{\mu c^2\alpha^2}{2(2)^2}\right) \\
&= -\frac{\mu c^2\alpha^2}{2}\frac{73}{100}; \\
\langle \vec{L}^2\rangle &= \left|\frac{4}{5}e^{-iE_1 t/\hbar}\right|^2 (0) + \left|\frac{3i}{5}e^{-iE_2 t/\hbar}\right|^2 \cdot 2\hbar^2 \\
&= \frac{18\hbar^2}{25}; \\
\langle \vec{L}_z\rangle &= \left|\frac{4}{5}e^{-iE_1 t/\hbar}\right|^2 (0) + \left|\frac{3i}{5}e^{-iE_2 t/\hbar}\right|^2 \cdot \hbar \\
&= \frac{9\hbar}{25}.
\end{aligned}$$

Since all three operators commute with the Hamiltonian, their expectation values are independent of time.

10.5 For each problem, compute

$$a_0 = \frac{\hbar}{\mu c\alpha} = \frac{\hbar^2}{\mu e^2}; \qquad E_1 = -\frac{\mu c^2\alpha^2}{2}; \tag{1}$$

$$\frac{hc}{\lambda_{21}} = E_2 - E_1 = -\frac{\mu c^2\alpha^2}{2(2)^2} + \frac{\mu c^2\alpha^2}{2} = \frac{3}{4}\frac{\mu c^2\alpha^2}{2}.$$

The reduced mass is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

For point of comparison, the values for hydrogen are

$$a_0 = .529\,\text{Å}; \qquad E_1 = -13.6\,\text{eV}; \qquad \lambda_{21} = 1216\,\text{Å (UV)},$$

and $\mu_H = \frac{m_e m_p}{m_e + m_p} = 9.104 \cdot 10^{-28}$ g, as compared to $m_e = 9.109 \cdot 10^{-28}$ g (to four sig figs).

(a) ^2H: A deuteron is a bound state of a neutron and proton, with mass $m_D \approx 2m_p$. Thus $\mu_{^2H} = \frac{m_e \cdot 2m_p}{m_e + 2m_p} \approx 1.00027\mu_H$. The quantities in (1) differ from those for hydrogen by this small factor.

(b) Positronium is a bound state of an electron and position. $\mu_{pos} = m_e/2 \approx \mu_H/2$. The Bohr radius is doubled as compared to hydrogen and the ground state energy is halved. The wavelength is doubled, still in the UV.

(c) The muon is 207 times more massive than the electron. $\mu = 186\mu_H$. The wavelength is $\lambda_{21} = 6.5\,\text{Å}$, an x-ray.

10.7 The nuclear reaction changes $Z = 1$ to $Z = 2$ in the ground-state wavefunction

$$\psi(\vec{r}) = R_{10}(r)Y_{00}(\theta, \phi) = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}.$$

The amplitude to remain in the ground state is

$$\begin{aligned} {}_{He^3}\langle 1,0,0|1,0,0\rangle_{tritium} &= \int d^3\vec{r}\, \left(\frac{2^3}{\pi a_0^3}\right)^{1/2} e^{-2r/a_0} \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0} \\ &= \frac{2\sqrt{2}}{\pi a_0^3} \int d^3\vec{r}\, e^{-3r/a_0} \\ &= \frac{2\sqrt{2}}{\pi a_0^3}(4\pi)\left(\frac{2a_0^3}{27}\right) = \frac{16\sqrt{2}}{27}. \end{aligned}$$

The probability is $512/729 \approx .70$.

12.2 The spin states symmetric under exchange of the particles are the triplet of states of total spin 1, i.e the three states

$$|s, m\rangle = |1, 1\rangle,\ |1, 0\rangle,\ \text{or}\ |1, -1\rangle.$$

In class we used $|\chi\rangle_S$ to denote any of these three possible states. The state antisymmetric under particle exchange is the singlet of total spin 0,

$$|s, m\rangle = |0, 0\rangle,$$

which we denoted $|\chi\rangle_A$.

(a) As we more or less worked out in class, the ground state and first excited states are:

$$\begin{aligned} &\text{ground:} && |0\rangle_1|0\rangle_2|\chi\rangle_A, \quad E_0 = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2} = \hbar\omega; \\ &\text{first excited:} && \frac{1}{\sqrt{2}}\Big(|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\Big)|\chi\rangle_A \\ & && \textit{or} \quad \frac{1}{\sqrt{2}}\Big(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\Big)|\chi\rangle_S, \quad E_1 = \frac{\hbar\omega}{2} + \frac{3\hbar\omega}{2} = 2\hbar\omega. \end{aligned}$$

(b) The spin singlet has a symmetric spatial wavefunction; the spin triplet has an antisymmetric spatial wavefunction. The two particles are more likely to be found near each other when they have a symmetric spatial wavefunction. Since the potential is negative when $|x_1 - x_2| < a$, we expect the correction to the energy from their interaction to be more negative when they have a symmetric spatial wavefunction. For all of the states, the interaction will lower the total energy, but moreso for the spin singlets than the spin triplet.