Phys 125
Boyda

Solutions to problems from Chapter 1.

1.3(b),(c) Beginning in the state

$$|+\boldsymbol{n}\rangle = \cos\frac{\theta}{2}|+\boldsymbol{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\boldsymbol{z}\rangle,$$

the probabilities for measurements of $\hat{S}_z$ are:

$$\begin{aligned} Prob\left(S_z \to +\frac{\hbar}{2}\right) &= |\langle +\boldsymbol{z}|+\boldsymbol{n}\rangle|^2 = \left|\cos\frac{\theta}{2}\right|^2 = \cos^2\frac{\theta}{2} \\ Prob\left(S_z \to -\frac{\hbar}{2}\right) &= |\langle -\boldsymbol{z}|+\boldsymbol{n}\rangle|^2 = \left|e^{i\phi}\sin\frac{\theta}{2}\right|^2 = \sin^2\frac{\theta}{2} \end{aligned}$$

Given this,

$$\begin{aligned} \langle S_z\rangle &= Prob\left(S_z \to +\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) + Prob\left(S_z \to -\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right) \\ &= \cos^2\frac{\theta}{2}\left(\frac{\hbar}{2}\right) + \sin^2\frac{\theta}{2}\left(-\frac{\hbar}{2}\right) \\ &= \frac{\hbar}{2}\cos\theta, \end{aligned}$$

while

$$\begin{aligned} \langle S_z^2\rangle &= Prob\left(S_z \to +\frac{\hbar}{2}\right)\left(\frac{\hbar^2}{4}\right) + Prob\left(S_z \to -\frac{\hbar}{2}\right)\left(\frac{\hbar^2}{4}\right) \\ &= \frac{\hbar^2}{4}. \end{aligned}$$

Thus

$$\begin{aligned} \Delta S_z &= \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} \\ &= \sqrt{\frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\cos\theta\right)^2} \\ &= \frac{\hbar}{2}\sin\theta. \end{aligned}$$

If $\theta = 0$, the state $|+\boldsymbol{n}\rangle = |+\boldsymbol{z}\rangle$ and the uncertainty for measurements of S_z is correspondingly zero. Similarly, if $\theta = \pi$, $|+\boldsymbol{n}\rangle = e^{i\phi}|-\boldsymbol{z}\rangle \sim |-\boldsymbol{z}\rangle$

up to an irrelevant overall phase, and again the outcome of the S_z measurment is perfectly certain. On the other hand, if $\theta = \pi/2$, the uncertainty is maximal, $\Delta S_z = \frac{\hbar}{2}$. In this case a measurement of S_z will yield $\pm\frac{\hbar}{2}$ as a 50/50 proposition, a swing of $\frac{\hbar}{2}$ either direction around the mean. The states with $\theta = \pi/2$ have spin lying in the $x - y$ plane.

1.5. This takes some care in the complex algebra.

$$\begin{aligned}\langle +\boldsymbol{y}|+\boldsymbol{n}\rangle &= \left(\frac{1}{\sqrt{2}}\langle +\boldsymbol{z}| - \frac{i}{\sqrt{2}}\langle -\boldsymbol{z}|\right)\left(\cos\frac{\theta}{2}|+\boldsymbol{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\boldsymbol{z}\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} - ie^{i\phi}\sin\frac{\theta}{2}\right).\end{aligned}$$

To get the probability, we do

$$\begin{aligned}|\langle +\boldsymbol{y}|+\boldsymbol{n}\rangle|^2 &= \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} - ie^{i\phi}\sin\frac{\theta}{2}\right)^* \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} - ie^{i\phi}\sin\frac{\theta}{2}\right)\\ &= \frac{1}{2}\left(\cos\frac{\theta}{2} + ie^{-i\phi}\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - ie^{i\phi}\sin\frac{\theta}{2}\right)\\ &= \frac{1}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - i\left(e^{i\phi} - e^{-i\phi}\right)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)\\ &= \frac{1}{2}(1 + \sin\theta\sin\phi)\end{aligned}$$

In the last line I used $\left(e^{i\phi} - e^{-i\phi}\right) = 2i\sin\phi$ and $\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{1}{2}\sin\theta$. You can check e.g., that for the case $\theta = 0$, corresponding to initial state $|+\boldsymbol{z}\rangle$, the probability is $\frac{1}{2}$ as expected. For $\theta = \phi = \pi/2$, corresponding to initial state $|+\boldsymbol{y}\rangle$, the probability is one.

Part (b). Since $\langle +\boldsymbol{n}|+\boldsymbol{y}\rangle = \langle +\boldsymbol{y}|+\boldsymbol{n}\rangle^*$, the probability (complex magnitude squared) must work out to be the same as in part (a).

1.7. By the first device the particles are prepared in state $|+\boldsymbol{z}\rangle$. The second device projects to particles with spin up along $\boldsymbol{n}$, with probability

$$|\langle +\boldsymbol{n}|+\boldsymbol{z}\rangle|^2 = \cos^2\frac{\theta}{2}.$$

Subsequently they are in state $|+\boldsymbol{n}\rangle$, to be measured along $\boldsymbol{z}$. The measurement along $\boldsymbol{z}$ yields spin down with probability

$$\begin{aligned}|\langle -\boldsymbol{z}|+\boldsymbol{n}\rangle|^2 &= \left|-e^{i\phi}\sin\frac{\theta}{2}\right|^2\\ &= \sin^2\frac{\theta}{2}.\end{aligned}$$

The fraction of particles finally transmitted is thus $\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{1}{4}\sin^2\theta$.

Part (b). The fraction is maximized for $\theta = \pi/2$, for which the $\boldsymbol{n}$ direction lies in the $x - y$ plane.

Part (c). Without the intermediate SG**n** device, we would have $|+\boldsymbol{z}\rangle$ particles entering the final SG**z** device, and zero particles would exit spin down. The intermediate selective measurement (as in part (b)) actually allows more particles to get through.

1.14. To begin, write the most general possible state in your favorite basis. Given the phrasing of the problem, the $|\pm\boldsymbol{z}\rangle$ basis is a good choice:

$$|\psi\rangle = c_+|+\boldsymbol{z}\rangle + c_-|-\boldsymbol{z}\rangle.$$

Note that $c_\pm$ are complex numbers with both magnitude *and* phase. The probabilities of finding the particle spin up/down along $\boldsymbol{z}$ fixes the magitudes $|c_+|^2 = .36$ and $|c_-|^2 = .64$. This allows us to write

$$|\psi\rangle = .6e^{i\phi_+}|+\boldsymbol{z}\rangle + .8e^{i\phi_-}|-\boldsymbol{z}\rangle,$$

for unknown phases ϕ_+ and ϕ_-. The final bit of information gives the probability of finding the particle with spin up along $\boldsymbol{x}$. First let's compute the amplitude:

$$\begin{aligned}
\langle +\boldsymbol{x}|\psi\rangle &= \left(\frac{1}{\sqrt{2}}\langle +\boldsymbol{z}| + \frac{1}{\sqrt{2}}\langle -\boldsymbol{z}|\right)\left(.6e^{i\phi_+}|+\boldsymbol{z}\rangle + .8e^{i\phi_-}|-\boldsymbol{z}\rangle\right)\\
&= \frac{1}{\sqrt{2}}\left(.6e^{i\phi_+} + .8e^{i\phi_-}\right).
\end{aligned}$$

The probability is then

$$\begin{aligned}
.5 &= \frac{1}{\sqrt{2}}\left(.6e^{i\phi_+} + .8e^{i\phi_-}\right)^* \frac{1}{\sqrt{2}}\left(.6e^{i\phi_+} + .8e^{i\phi_-}\right)\\
&= \frac{1}{2}\left(.36 + .64 + .48(e^{i(\phi_+ - \phi_-)} + e^{-i(\phi_+ - \phi_-)})\right)\\
&= \frac{1}{2}(1 + .96\cos(\phi_+ - \phi_-)).
\end{aligned}$$

The argument of the cosine, $(\phi_+ - \phi_-)$, must equal $\pm\frac{\pi}{2}$, resulting in

$$\begin{aligned}
|\psi\rangle &= .6e^{i\phi_+}|+\boldsymbol{z}\rangle + .8e^{i\phi_-}|-\boldsymbol{z}\rangle\\
&= e^{i\phi_+}\left(.6|+\boldsymbol{z}\rangle + .8e^{i(\phi_- - \phi_+)}|-\boldsymbol{z}\rangle\right)\\
&= e^{i\phi_+}\left(.6|+\boldsymbol{z}\rangle \pm .8i|-\boldsymbol{z}\rangle\right).
\end{aligned}$$

That's as far as you can go. The $+\boldsymbol{x}$ measurement (random probability 50/50 of being spin up/down along $\boldsymbol{x}$) implies that the spin lies in the $y - z$ plane. It is closer to $|-\boldsymbol{z}\rangle$ than $|+\boldsymbol{z}\rangle$ (probability .64 vs. .36), but we have no information as to whether the spin, thus inclined, points toward the side of $|+\boldsymbol{y}\rangle$ or $|-\boldsymbol{y}\rangle$. (In speaking of the spin as "pointing" in a certain direction, we have to remember that this is really the direction about which the particle undertakes a "fuzzy precession." With the spin lying in the $y - z$ plane, there is still always, for instance, a 50% chance that an S_x measurement will find spin-up along $\boldsymbol{x}$.