Phys 125
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Solutions to problems from Chapter 2 (along with 1.10).

1.10 In the usual spherical coordinates, a vector on the unit sphere is given by

$$\vec{n} = \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k}.$$

The spin oriented up along this axis has state

$$|+\boldsymbol{n}\rangle = \cos\frac{\theta}{2}|\boldsymbol{z}+\rangle + e^{i\phi}\sin\frac{\theta}{2}|\boldsymbol{z}-\rangle,$$

and for angles $\theta = 2\pi/3$, $\phi = \pi/2$, this becomes the desired state

$$|\psi = \frac{1}{2}|\boldsymbol{z}+\rangle + \frac{i\sqrt{3}}{2}|\boldsymbol{z}-\rangle.$$

The corresponding unit vector in three-dimesional space is

$$\vec{n} = \frac{\sqrt{3}}{2}\hat{\jmath} - \frac{1}{2}\hat{k}.$$

Since we are working on matrix notation, let's compute an expectation value in matrix notation in the $|\boldsymbol{z}\pm\rangle$ basis. The matrix corresponding to $\hat{S}_y$ is

$$\hat{S}_y \to \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We compute:

$$\begin{aligned}
\langle J_y\rangle = \langle\psi|\hat{J}_y|\psi\rangle &\to \begin{pmatrix} \frac{1}{2} & \frac{-i\sqrt{3}}{2} \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{i\sqrt{3}}{2} \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \frac{1}{2} & \frac{-i\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{i}{2} \end{pmatrix} \\
&= \frac{\hbar}{2}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right) \\
&= \frac{\sqrt{3}}{4}.
\end{aligned}$$

By similar method you can compute

$$\langle J_x\rangle = 0;$$
$$\langle J_z\rangle = -\frac{\hbar}{4}.$$

The axis along which the spin is oriented points largely along the $+\hat{\jmath}$ direction, with a smaller component along $-\hat{k}$.

2.2 The projection operator $\hat{P}_+ = |\boldsymbol{z}+\rangle\langle\boldsymbol{z}+|$ is Hermtian on the face of it, and that guarantees the existence of a basis of eigenvectors. Supposing $|v\rangle$ to be one such eigenvector with eigenvalue λ,

$$\hat{P}_+|v\rangle = \lambda|v\rangle, \quad \text{and also,}$$
$$\hat{P}_+|v\rangle = P_+^2|v\rangle = \hat{P}_+\lambda|v\rangle = \lambda\hat{P}_+|v\rangle = \lambda^2|v\rangle$$

In consequence, $\lambda|v\rangle = \lambda^2|v\rangle$, which implies $\lambda = 0$ or 1. In general for a projection operator, the vector onto which the operator projects has eigenvalue one (it is sent to itself under the projection), and vectors orthogonal to it have eigenvalue zero (they are projected to the zero vector).

2.5 The matrix elements A_{ij} of an operator $\hat{A}$ in basis $\{|e_i\rangle\}$ are given by

$$A_{ij} = \langle e_i|\hat{A}|e_j\rangle.$$

The i indexes the row and the j indexes the column of the matrix. In this problem, we are given basis $\{|\boldsymbol{y}+\rangle, |\boldsymbol{y}-\rangle\}$ and asked to find matrix elements of $\hat{J}_z$. Assembled as a matrix, they are

$$\begin{pmatrix} \langle\boldsymbol{y}+|\hat{J}_z|\boldsymbol{y}+\rangle & \langle\boldsymbol{y}+|\hat{J}_z|\boldsymbol{y}-\rangle \\ \langle\boldsymbol{y}-|\hat{J}_z|\boldsymbol{y}+\rangle & \langle\boldsymbol{y}-|\hat{J}_z|\boldsymbol{y}-\rangle \end{pmatrix}.$$

The individual elements can be computed as follows:

$$\begin{aligned} \langle\boldsymbol{y}+|\hat{J}_z|\boldsymbol{y}+\rangle &= \frac{1}{\sqrt{2}}\left(\langle\boldsymbol{z}+| - i\langle\boldsymbol{z}-|\right)\left(\frac{\hbar}{2}|\boldsymbol{z}+\rangle\langle\boldsymbol{z}+| - \frac{\hbar}{2}|\boldsymbol{z}-\rangle\langle\boldsymbol{z}-|\right)\frac{1}{\sqrt{2}}\left(|\boldsymbol{z}+\rangle + i|\boldsymbol{z}-\rangle\right) \\ &= \frac{1}{2}\left(\frac{\hbar}{2} - \frac{\hbar}{2}\right) = 0; \\ \langle\boldsymbol{y}+|\hat{J}_z|\boldsymbol{y}-\rangle &= \frac{1}{\sqrt{2}}\left(\langle\boldsymbol{z}+| - i\langle\boldsymbol{z}-|\right)\left(\frac{\hbar}{2}|\boldsymbol{z}+\rangle\langle\boldsymbol{z}+| - \frac{\hbar}{2}|\boldsymbol{z}-\rangle\langle\boldsymbol{z}-|\right)\frac{1}{\sqrt{2}}\left(|\boldsymbol{z}+\rangle - i|\boldsymbol{z}-\rangle\right) \\ &= \frac{1}{2}\left(\frac{\hbar}{2} + \frac{\hbar}{2}\right) = \frac{\hbar}{2}; \end{aligned}$$

etc.

Altogether, in the $|\boldsymbol{y}\pm\rangle$ basis,

$$\hat{J}_z \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is the same as the matrix for J_x in the $|\boldsymbol{z}\pm\rangle$ basis. That it must be so can be seen by permuting the labels x, y, z.

In the $|\boldsymbol{y}\pm\rangle$ basis the state $|\boldsymbol{y}-\rangle$ has column vector representation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the expectation value of J_z in this state is

$$\langle J_z\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\hbar}{2} \\ 0 \end{pmatrix} = 0.$$

2.7 As in the previous problem, we begin with the matrix in form

$$\begin{pmatrix} \langle \boldsymbol{y}+|\hat{P}_+|\boldsymbol{y}+\rangle & \langle \boldsymbol{y}+|\hat{P}_+|\boldsymbol{y}-\rangle \\ \langle \boldsymbol{y}-|\hat{P}_+|\boldsymbol{y}+\rangle & \langle \boldsymbol{y}-|\hat{P}_+|\boldsymbol{y}-\rangle \end{pmatrix},$$

where $\hat{P}_+ = |\boldsymbol{z}+\rangle\langle\boldsymbol{z}+|$. You just have to compute matrix elements of $|\boldsymbol{z}+\rangle$ with $\langle \boldsymbol{y}\pm|$, e.g.,

$$\langle \boldsymbol{y}+|\hat{P}_+|\boldsymbol{y}+\rangle = \langle \boldsymbol{y}+|\boldsymbol{z}+\rangle\langle\boldsymbol{z}+|\boldsymbol{y}+\rangle = \left(\frac{i}{\sqrt{2}}\right)^* \left(\frac{i}{\sqrt{2}}\right) = \frac{1}{2}.$$

You find:

$$\begin{aligned} P_+ =& \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ P_- =& \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

These are projectors, e.g.,

$$\begin{aligned} P_-^2 =& \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \\ &= P_-. \end{aligned}$$

Since they project onto vectors that are orthogonal to one another, following one by the other projects to zero:

$$\begin{aligned} P_-P_+ =& \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and similarly for P_+P_-.

2.8 The state

$$|\psi\rangle \to \frac{1}{\sqrt{5}}\begin{pmatrix} i \\ 2 \end{pmatrix}$$

is normalized:

$$\langle\psi|\psi\rangle \to \frac{1}{\sqrt{5}}\begin{pmatrix} -i & 2 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} i \\ 2 \end{pmatrix} = \frac{1}{5}(1+4) = 1.$$

The requested probabilities are

$$Prob(S_x \to \frac{\hbar}{2}) = |\langle \boldsymbol{x+}\, |\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} i \\ 2 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} (i+2) \right|^2 = \frac{1}{2};$$

$$Prob(S_y \to \frac{\hbar}{2}) = |\langle \boldsymbol{y+}\, |\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} i \\ 2 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} (i-2i) \right|^2 = \frac{1}{10};$$