Phys 125
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Solutions to later problems from Chapter 2. Photons!

2.11 Before we get to photons, we have one problem with spin-1/2 particles. We are given, in the S_z basis,

$$|\psi\rangle \to \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}.$$

We know that in this basis,

$$\hat{S}_x \to \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Therefore,

$$\langle\psi|\hat{S}_x|\psi\rangle \to \frac{1}{\sqrt{3}}\begin{pmatrix}1 & \sqrt{2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix} = \frac{\hbar}{6}\begin{pmatrix}1 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2}\\ 1\end{pmatrix} = \frac{\hbar\sqrt{2}}{3}.$$

2.12 We are given

$$|\psi\rangle = \sqrt{\frac{2}{3}}|X\rangle + \frac{i}{\sqrt{3}}|Y\rangle.$$

(a)

$$|\langle Y|\psi\rangle|^2 = \left|\frac{i}{\sqrt{3}}\right|^2 = \frac{1}{3}.$$

(b) Use the rotation operator around $\hat{z}$ to find the new state from the state $|Y\rangle$. This is most compactly written in matrix notation, so we'll switch back and forth between bra-ket and matrices. In the $X - Y$ basis the state $|Y\rangle$ is $\begin{pmatrix}0\\ 1\end{pmatrix}$.

$$|Y'\rangle = \hat{R}(\theta\hat{z})|Y\rangle \to \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}, \quad \text{i.e.}$$
$$|Y'\rangle = -\sin\theta|X\rangle + \cos\theta|Y\rangle.$$

Then,

$$|\langle Y'|\psi\rangle|^2 = \left|-\sqrt{\frac{2}{3}}\sin\theta + \frac{i}{\sqrt{3}}\cos\theta\right|^2 = \frac{2}{3}\sin^2\theta + \frac{1}{3}\cos^2\theta = \frac{1}{3}(1+\sin^2\theta).$$

(c) Torque in general is given by the rate of change of angular momentum, $\vec{\tau} = d\vec{L}/dt$. The average torque is the average angular momentum per photon times the number N of photons per second impacting the black disk, so here,

$$\tau_z = \langle \hat{l}_z \rangle N.$$

We have $|\psi\rangle$ in the $X - Y$ basis but we are most familiar with $\hat{l}_z$ in the $R - L$ basis, where $\hat{l}_z = \hbar|R\rangle\langle R| - \hbar|L\rangle\langle L|$. We can compute in bra-ket notation as follows:

$$\langle \hat{l}_z \rangle = \langle \psi|\hat{l}_z|\psi\rangle = \left(\sqrt{\frac{2}{3}}\langle X| + \frac{-i}{\sqrt{3}}\langle Y|\right)(\hbar|R\rangle\langle R| - \hbar|L\rangle\langle L|)\,|\left(\sqrt{\frac{2}{3}}|X\rangle + \frac{i}{\sqrt{3}}|Y\rangle\right).$$

I leave this for you as an exam practice problem. Another method is to find the matrix representation of $\hat{l}_z$ in the $X - Y$ basis. It is given in Example 2.8 as

$$\hat{l}_z \to \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (X - Y \text{ basis}).$$

In this representation,

$$\begin{aligned} \langle \psi|\hat{l}_z|\psi\rangle \to \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{-i}{\sqrt{3}} \end{pmatrix} \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{i}{\sqrt{3}} \end{pmatrix} \\ \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{-i}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{\hbar}{\sqrt{3}} \\ i\hbar\sqrt{\frac{2}{3}} \end{pmatrix} \\ = \frac{2\hbar\sqrt{2}}{3}. \end{aligned}$$

Thus,

$$\tau_z = \frac{2\hbar N\sqrt{2}}{3}.$$

(d) The result from (a) is unchanged, and in (b) there is an additional cross term picked up. Physically speaking, the initial $|\psi\rangle$ is an elliptical polarization, but by removing the i, the state is made to be wholly linear in polarization. In this new state there is equal probability of finding $|R\rangle$ and $|L\rangle$ circular polarizations, as you can check (more exam practice). That means $\langle \hat{l}_z \rangle = 0$, and so the torque is also zero.

2.14 An arbitrary state $|\psi\rangle$ can be written in the $X - Y$ basis by applying the identity operator in the form $\hat{I} = |X\rangle\langle X| + |Y\rangle\langle Y|$:

$$|\psi\rangle = |X\rangle\langle X|\psi\rangle + |Y\rangle\langle Y|\psi\rangle.$$

The coefficients multiplying the basis vectors are $c_X = \langle X|\psi\rangle$ and $c_Y = \langle Y|\psi\rangle$. In the matrix representation, then,

$$|\psi\rangle \to \begin{pmatrix} \langle X|\psi\rangle \\ \langle Y|\psi\rangle \end{pmatrix}.$$

Expressing it this way can help you get the basis transformation straight. We want to end up with a representation in the $R-L$ basis, in which

$$|\psi\rangle \to \begin{pmatrix} \langle R|\psi\rangle \\ \langle L|\psi\rangle \end{pmatrix}.$$

The required transformation is

$$\begin{aligned} \begin{pmatrix} \langle R|\psi\rangle \\ \langle L|\psi\rangle \end{pmatrix} &= \begin{pmatrix} \langle R|X\rangle & \langle R|Y\rangle \\ \langle L|X\rangle & \langle L|Y\rangle \end{pmatrix} \begin{pmatrix} \langle X|\psi\rangle \\ \langle Y|\psi\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle R|X\rangle\langle X|\psi\rangle + \langle R|Y\rangle\langle Y|\psi\rangle \\ \langle L|X\rangle\langle X|\psi\rangle + \langle L|Y\rangle\langle Y|\psi\rangle \end{pmatrix}. \end{aligned}$$

Hopefully you can see that this transformation amounts, in each line, to the application of the same identity operator. Computing out the brackets, the transformation matrix is

$$\mathbb{S} = \begin{pmatrix} \langle R|X\rangle & \langle R|Y\rangle \\ \langle L|X\rangle & \langle L|Y\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

To show that this is unitary we check that $\mathbb{S}^\dagger\mathbb{S} = I$ and $\mathbb{S}\mathbb{S}^\dagger = I$, both, because matices don't commute in general and a right-side inverse is not necessarily a left-side inverse.

$$\begin{aligned} \mathbb{S}\mathbb{S}^\dagger &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I; \\ \mathbb{S}^\dagger\mathbb{S} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I. \end{aligned}$$

2.23 (a) We need to check that states $|a'_n\rangle = U|a_n\rangle$ are normalized to one and that they are mutually orthogonal ($0 = \langle a'_n|a'_m\rangle$ when $n \neq m$). We use the fact that $\hat{U}^\dagger\hat{U} = \hat{I}$ and handle both cases ($n = m$ or $n \neq m$) compactly,

$$\langle a'_n|a'_m\rangle = \langle a_n|\hat{U}^\dagger\hat{U}|a_m\rangle = \langle a_n|a_m\rangle = \delta_{nm}.$$

(b) For an eigenstate $|n\rangle$, $\hat{U}|n\rangle = \lambda_n|n\rangle$ for some eigenvalue λ_n. Then

$$\langle n|n\rangle = \langle n|\hat{U}^\dagger\hat{U}|n\rangle = \lambda_n^*\lambda_n\langle n|n\rangle.$$

Therefore $\lambda_n^*\lambda_n = 1$, and λ_n is a complex number of magnitude one: $\lambda_n = e^{i\phi_n}$.