Phys 125
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Solutions to problems from Chapter 3.

3.15 We need to find spin-1 eigenstates of $\hat{S}_x$ in the basis of $\hat{S}_z$ eigenstates. The matrix representation of $\hat{S}_x$ in the S_z basis is

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We already know the eigenvalues, so we have a leg up in this game. They are $\hbar$, 0, $-\hbar$. To mix things up, let's first find the state $|1,-1\rangle_x$, for which

$$\hat{S}_x|1,-1\rangle_x = -\hbar|1,-1\rangle_x.$$

Allowing for arbitrary vector coefficients a, b, c, we set out to solve

$$S_x \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = -\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Whence, $c = a = -b/\sqrt{2}$, and our vector becomes

$$\begin{pmatrix} -\frac{b}{\sqrt{2}} \\ b \\ -\frac{b}{\sqrt{2}} \end{pmatrix}.$$

We normalize to find b:

$$1 = \begin{pmatrix} -\frac{b^*}{\sqrt{2}} & b^* & -\frac{b^*}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{b}{\sqrt{2}} \\ b \\ -\frac{b}{\sqrt{2}} \end{pmatrix} = 2|b|^2 \quad \rightarrow \quad b = \frac{1}{\sqrt{2}}.$$

The result is

$$|1,-1\rangle_x \rightarrow \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}.$$

You can similarly work out

$$|1,0\rangle_x \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \qquad |1,1\rangle_x \rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

3.16 We use the result from 3.15. The amplitude is

$$_x\langle 1,0|1,1\rangle_z = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}},$$

so the probability is one half.

3.17 We are given, in S_z basis,

$$|\psi\rangle \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3i\end{pmatrix}.$$

(a) We read off the probabilities 1/14, 4/14, 9/14. Multiplying probabilities times possible outcomes for measurements of S_z,

$$\langle S_z\rangle = \frac{1}{14}\cdot\hbar + \frac{4}{14}\cdot 0 + \frac{9}{14}\cdot(-\hbar) = -\frac{4\hbar}{7}.$$

(b) The matrix rep for S_x is given above.

$$\begin{aligned}\langle S_z\rangle &= \frac{1}{\sqrt{14}}\begin{pmatrix}1 & 2 & -3i\end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix}\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3i\end{pmatrix}\\ &= \frac{\hbar}{14\sqrt{2}}\begin{pmatrix}1 & 2 & -3i\end{pmatrix}\begin{pmatrix}2\\1+3i\\2\end{pmatrix} = \frac{\sqrt{2}\hbar}{7}.\end{aligned}$$

(c) The amplitude is

$$_x\langle 1,1|\psi\rangle = \frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3i\end{pmatrix} = \frac{1+2\sqrt{2}+3i}{2\sqrt{14}},$$

so the probability is $(9+2\sqrt{2})/28$.

3.20 We need first of all to find the eigenstates of the $\hat{S}_n$ operator, where $\hat{n}$ lies at angle θ from the $\hat{z}$ axis in the x-z plane. As the hint suggests, they can be found by applying the rotation operator $\hat{R}(\theta\hat{y})$, given in problem 3.19, to the eigenstates of $\hat{S}_z$. Since we are asked questions and given the $\hat{R}(\theta\hat{y})$ operator with respect to the basis of $\hat{S}_z$ eigenstates, that is the convient basis to work in. We find:

$$S_n = \hbar: \qquad \hat{R}(\theta\hat{y})\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{pmatrix}$$

$$S_n = 0: \qquad \hat{R}(\theta\hat{y})\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-\frac{\sin\theta}{\sqrt{2}}\\ \cos\theta\\ \frac{\sin\theta}{\sqrt{2}}\end{pmatrix}$$

$$S_n = -\hbar: \qquad \hat{R}(\theta\hat{y})\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{1-\cos\theta}{2}\\ -\frac{\sin\theta}{\sqrt{2}}\\ \frac{1+\cos\theta}{2}\end{pmatrix}$$

You can then compute amplitudes as

$$_n\langle 1,1|1,1\rangle_z = \begin{pmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1+\cos\theta}{2};$$

$$_z\langle 1,-1|1,1\rangle_n = \begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{pmatrix} = \frac{1-\cos\theta}{2};$$

$$_z\langle 1,0|1,1\rangle_n = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1+\cos\theta}{2}\\ \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2}\end{pmatrix} = \frac{\sin\theta}{\sqrt{2}}.$$

(a) Those particles that pass the first SG$\hat{z}$ apparatus are in state $|1,1\rangle_z$, and after the second apparatus they are in $|1,1\rangle_n$. The probability of passing through the SG sequence is:

$$\begin{aligned}|_z\langle 1,-1|1,1\rangle_n|^2\,|_n\langle 1,1|1,1\rangle_z|^2 &= \left|\frac{1-\cos\theta}{2}\right|^2\left|\frac{1+\cos\theta}{2}\right|^2\\ &= \left|\frac{1-\cos^2\theta}{4}\right|^2\\ &= \frac{\sin^4\theta}{16}.\end{aligned}$$

(b) The function $\sin^4\theta/16$ has maximum value 1/16 for $\theta = \pi/2$. This corresponds to $\hat{n} = \hat{x}$. The probabilities $|_x\langle 1,1|1,1\rangle_z|^2$ and $|_z\langle 1,-1|1,1\rangle_x|^2$ are 1/4 each.

(c) Zero. The states $|1,1\rangle_z$ and $|1,-1\rangle_z$ are orthogonal.

(d) Repeating for a final measurement of $S_z = 0$, we find probability of transmission

$$\begin{aligned}\left|{}_z\langle 1,0|1,1\rangle_n\right|^2 \left|{}_n\langle 1,1|1,1\rangle_z\right|^2 &= \left|\frac{\sin\theta}{\sqrt{2}}\right|^2 \left|\frac{1+\cos\theta}{2}\right|^2 \\ &= \frac{\sin^2\theta(1+\cos\theta)^2}{8}.\end{aligned}$$

To maximize this function, take the derivative with respect to θ and set to zero:

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{1}{8}\sin^2\theta(1+\cos\theta)^2\right) =& 2\sin\theta\cos\theta(1+\cos\theta)^2 - 2\sin^3\theta(1+\cos\theta) = 0 \\ & \cos\theta + \cos^2\theta - \sin^2\theta = 0 \\ & \cos\theta + \cos 2\theta = 0 \\ & \theta = \frac{\pi}{3}.\end{aligned}$$

The max transmission probability is $27/128$ at $\theta = \pi/3$, greater by factor ~ 3 from the previous result. In this very specific sense involving manipulation by an intermediate SG device, $S_z = 0$ is "closer" to $S_z = \hbar$ than is $S_z = -\hbar$. However, if you remove the intermediate SG$\hat{n}$ device, the transmission probability remains zero, because $|1,1\rangle_z$ and $|1,0\rangle_z$ are strictly orthogonal.

3.21 For a spin-j particle, $|\vec{J}| = \sqrt{j(j+1)}\hbar$. We have talked about the quantum particle in the z-spin-up state undergoing "fuzzy" or "smeared-out" precession about the z axis. Without taking this angle $\theta = \arccos J_z/|\vec{J}|$ too literally, we can use it as a way of thinking about how far the z-spin-up state is tipped from the z axis.

$$\begin{aligned} j = \frac{1}{2}: &\qquad \theta = \arccos\frac{\frac{\hbar}{2}}{\frac{\sqrt{3}\hbar}{2}} \approx 55^\circ \\ j = 1: &\qquad \theta = \arccos\frac{\hbar}{\sqrt{2}\hbar} = \pi/4.\end{aligned}$$

In the macroscopic limit, this angle tends to zero.

3.22 We need the state $|\frac{3}{2},\frac{1}{2}\rangle_x$ in the S_z basis. It is given in problem 3.23 as

$$|\frac{3}{2},\frac{1}{2}\rangle_x \rightarrow \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\1\\-1\\-\sqrt{3}\end{pmatrix}.$$

From there we read off amplitudes,

$$ {}_z\langle\frac{3}{2},\frac{3}{2}|\frac{3}{2},\frac{1}{2}\rangle_x = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\1\\-1\\-\sqrt{3}\end{pmatrix} = \frac{\sqrt{3}}{2\sqrt{2}}; $$

$$ {}_z\langle\frac{3}{2},\frac{1}{2}|\frac{3}{2},\frac{1}{2}\rangle_x = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\1\\-1\\-\sqrt{3}\end{pmatrix} = \frac{1}{2\sqrt{2}}; $$

$$ {}_z\langle\frac{3}{2},-\frac{1}{2}|\frac{3}{2},\frac{1}{2}\rangle_x = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}\frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\1\\-1\\-\sqrt{3}\end{pmatrix} = \frac{-1}{2\sqrt{2}}; $$

$$ {}_z\langle\frac{3}{2},-\frac{3}{2}|\frac{3}{2},\frac{1}{2}\rangle_x = \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}\frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\1\\-1\\-\sqrt{3}\end{pmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}}. $$

Squaring these values gives probabilities $\frac{3}{8}$, $\frac{1}{8}$, $\frac{1}{8}$, $\frac{3}{8}$.

3.23

$$ \frac{\hbar}{2}\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}\frac{1}{2\sqrt{2}}\begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{3}\\ 1\end{pmatrix} = \frac{\hbar}{2}\frac{1}{2\sqrt{2}}\begin{pmatrix}3\\ 3\sqrt{3}\\ 3\sqrt{3}\\ 3\end{pmatrix} = \frac{3\hbar}{2}\frac{1}{2\sqrt{2}}\begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{3}\\ 1\end{pmatrix}, $$

so this is an eigenstate of $\hat{S}_x$ with eigenvalue $3\hbar/2$. As you can check, the four states form an orthonormal basis.

3.24 (a) Normalize:

$$ 1 = \langle\psi|\psi\rangle = N^*\begin{pmatrix}-i & 2 & 3 & -4i\end{pmatrix}N\begin{pmatrix}i\\2\\3\\4i\end{pmatrix} = 30|N|^2 \quad\Rightarrow\quad N = \frac{1}{\sqrt{30}}. $$

(b) Using the expression for $\hat{S}_x$ in the S_z basis given in problem 3.23, we compute in matrix notation,

$$\langle\psi|\hat{S}_x|\psi\rangle \to \frac{1}{\sqrt{30}}\begin{pmatrix}-i & 2 & 3 & -4i\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix}i\\ 2\\ 3\\ 4i\end{pmatrix}$$

$$= \frac{\hbar}{60}\begin{pmatrix}-i & 2 & 3 & -4i\end{pmatrix}\begin{pmatrix}2\sqrt{3}\\ i\sqrt{3}+6\\ 4+4i\sqrt{3}\\ 3\sqrt{3}\end{pmatrix}$$

$$= \frac{\hbar}{60}(-2i\sqrt{3}+2i\sqrt{3}+12+12+12i\sqrt{3}-12i\sqrt{3})$$

$$= \frac{2\hbar}{5}$$

Since $\hat{S}_x$ is Hermitian, this must be real. You can use this to check your computation.

(c)

$$_x\langle\frac{3}{2},\frac{1}{2}|\psi\rangle \to \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3} & 1 & -1 & -\sqrt{3}\end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix}i\\ 2\\ 3\\ 4i\end{pmatrix} = \frac{1}{2\sqrt{60}}(i\sqrt{3}+2-3-4i\sqrt{3}).$$

Upon squaring, the probability is 7/60.

3.26 We are told that $\hat{A}$ and $\hat{B}$ share a basis of eigenvectors $\{|e_i\rangle\}$, such that $\hat{A}|e_i\rangle = a_i|e_i\rangle$ and $\hat{B}|e_i\rangle = b_i|e_i\rangle$. It suffices to check the action of the operators on the basis vectors.

$$\hat{A}\hat{B}|e_i\rangle = \hat{A}b_i|e_i\rangle = b_i\hat{A}|e_i\rangle = b_ia_i|e_i\rangle = a_ib_i|e_i\rangle = a_i\hat{B}|e_i\rangle = \hat{B}a_i|e_i\rangle = \hat{B}\hat{A}|e_i\rangle.$$

Because we can commute the eigenvalues (numbers), we can commute the operators.