Phys 125
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Solutions to problems from Chapter 4.

For a time-independent Hamiltonian, the plan of attack is as follows:

1. Expand the initial state as a superposition of energy eigenstates.
2. Apply the time evolution operator $e^{-i\hat{H}t/\hbar}$ to each term in the superposition. It acts simply on energy eigenstates, the operator $\hat{H}$ being replaced by the relevant eigenvalue.
3. Take amplitudes and square.

The solutions below execute this basic strategy.

4.4 We are given $\hat{H} = \omega_0 \hat{S}_x$ and initial state $|\psi(0)\rangle = |z+\rangle$. We are asked to compute the time t for which

$$|\langle z - |\psi(t)\rangle|^2 = \frac{1}{4}.$$

The eigenstates of the Hamiltonian are $|x\pm\rangle$ with eigenvalues $\pm\hbar\omega_0/2$. We expand the initial state in this basis, $|z+\rangle = \frac{1}{\sqrt{2}}|x+\rangle + \frac{1}{\sqrt{2}}|x-\rangle$, whence

$$\begin{aligned}|\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}|z+\rangle = e^{-i\hat{H}t/\hbar}\left(\frac{1}{\sqrt{2}}|x+\rangle + \frac{1}{\sqrt{2}}|x-\rangle\right) \\ &= \frac{1}{\sqrt{2}}e^{-i\omega_0 t/2}|x+\rangle + \frac{1}{\sqrt{2}}e^{+i\omega_0 t/2}|x-\rangle.\end{aligned}$$

The amplitude is

$$\begin{aligned}\langle z - |e^{-i\hat{H}t/\hbar}|z+\rangle &= \langle z - | \left(\frac{1}{\sqrt{2}}e^{-i\omega_0 t/2}|x+\rangle + \frac{1}{\sqrt{2}}e^{+i\omega_0 t/2}|x-\rangle\right) \\ &= \frac{1}{2}e^{-i\omega_0 t/2} - \frac{1}{2}e^{+i\omega_0 t/2} \\ &= -i\sin\frac{\omega_0 t}{2}.\end{aligned}$$

In the intermediate step I used $\langle z - |x+\rangle = \frac{1}{\sqrt{2}}$ and $\langle z - |x-\rangle = -\frac{1}{\sqrt{2}}$. The time $t = l_0/v_0$.

$$\left|-i\sin\frac{\omega_0 t}{2}\right|^2 = \frac{1}{4} \quad \rightarrow \quad \frac{\omega_0 l_0}{2v_0} = \frac{\pi}{6} \quad \rightarrow \quad l_0 = \frac{v_0 \pi}{3\omega_0}.$$

4.5 For this problem we will use the basis given in problems 1.3 and 1.6 for spin up/down along $\hat{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$, but with $\phi = 0$ because the magnetic field lies in the x-z plane. The states are

$$
\begin{aligned}
|n+\rangle &= \cos\frac{\theta}{2}|z+\rangle + \sin\frac{\theta}{2}|z-\rangle \qquad (1)\\
|n-\rangle &= \sin\frac{\theta}{2}|z+\rangle - \cos\frac{\theta}{2}|z-\rangle.
\end{aligned}
$$

These states are the eigenvectors of the Hamlitonian $\hat{H} = \omega_0\hat{S}_n$, and the eigenvalues are $\pm\hbar\omega_0/2$. Expand the initial state $|z+\rangle$ in this basis to apply the time evolution operator:

$$
\begin{aligned}
e^{-i\hat{H}t/\hbar}|z+\rangle &= e^{-i\hat{H}t/\hbar}\left(\cos\frac{\theta}{2}|n+\rangle + \sin\frac{\theta}{2}|n-\rangle\right)\\
&= \cos\frac{\theta}{2}e^{-i\omega_0 t/2}|n+\rangle + \sin\frac{\theta}{2}e^{+i\omega_0 t/2}|n-\rangle.
\end{aligned}
$$

Using the formulas (1) and the known amplitudes $\langle y+|z+\rangle = 1/\sqrt{2}$ and $\langle y+|z-\rangle = -i/\sqrt{2}$, we can compute

$$
\begin{aligned}
\langle y+|n+\rangle &= \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}e^{-i\theta/2}\\
\langle y+|n-\rangle &= \frac{1}{\sqrt{2}}\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right) = i\frac{1}{\sqrt{2}}e^{-i\theta/2}
\end{aligned}
$$

The probability of finding spin-up through the SG$\hat{y}$ apparatus is

$$
\begin{aligned}
\left|\langle y+|e^{-i\hat{H}t/\hbar}|z+\rangle\right|^2 &= \left|\frac{1}{\sqrt{2}}e^{-i\theta/2}\cos\frac{\theta}{2}e^{-i\omega_0 t/2} + i\frac{1}{\sqrt{2}}e^{-i\theta/2}\sin\frac{\theta}{2}e^{+i\omega_0 t/2}\right|^2\\
&= \frac{\cos^2\frac{\theta}{2}}{2}\left|1 + i\tan\frac{\theta}{2}e^{i\omega_0 t}\right|^2\\
&= \frac{\cos^2\frac{\theta}{2}}{2}\left(1 + \left(i\tan\frac{\theta}{2}e^{i\omega_0 t} - i\tan\frac{\theta}{2}e^{-i\omega_0 t}\right) + \tan^2\frac{\theta}{2}\right)\\
&= \frac{\cos^2\frac{\theta}{2}}{2}\left(1 - 2\tan\frac{\theta}{2}\sin\omega_0 t + \tan^2\frac{\theta}{2}\right)\\
&= \frac{1}{2} - \frac{1}{2}\sin\theta\sin\omega_0 t.
\end{aligned}
$$

At $\theta = 0$ the magnetic field is along the $\hat{z}$ axis. Our initial state would be the stationary state $|z+\rangle$ (constant up to overall phase), and so the probability of finding spin-up along y would be $1/2$, independent of time. This checks out. At $\theta = \pi/2$, the magnetic field is along $\hat{x}$ and

$$
Prob(S_y \to +\frac{\hbar}{2}) \to \frac{1}{2} - \frac{1}{2}\sin\omega_0 t.
$$

The spin precesses $|z+\rangle \to |y-\rangle \to |z-\rangle \to |y+\rangle \to |z+\rangle$. I won't try to describe it, but try to picture for yourself the general case in which the spin precesses around the magnetic field at $0 < \theta < \pi/2$.

There are other ways to do this problem: You can work out the matrix of the evolution operator in the $|z\pm\rangle$ basis; or you can rotate to new coordinates x', y', z' in which the magnetic field is aligned with $\hat{z}'$ and the initial state is rotated away from $\hat{z}'$ in the x'-z' plane.

4.7 The mass of the muon is $m_\mu = 1.88 \cdot 10^{-25}\,\text{g}$, we are given a magnetic field of $60\,\text{G}$, and from the data we read a precession frequency $\nu = 807.5\,\text{kHz}$. We need to be consistent in our use of cgs units because the electromagnetic formulas differ between systems of units. Pertinent here are the fomulas for magnetic moment:

$$\vec{\mu} = \frac{gq}{2mc}\vec{S} \ \ (cgs) \quad \text{vs.} \quad \vec{\mu} = \frac{gq}{2m}\vec{S} \ \ (SI).$$

In cgs units the electric charge is $e = 4.80 \cdot 10^{-10}$ statcoulombs.

$$2\pi\nu = \omega_0 = \frac{g_\mu e B_0}{2m_\mu c} \quad \to g_\mu = \frac{4\pi(407.5 \cdot 10^3\,\text{Hz})(1.88 \cdot 10^{-25}\,\text{g})(3 \cdot 10^{10}\,\text{cm/s})}{(4.80 \cdot 10^{-10}\,\text{statcoulomb})(60\,\text{G})} = 1.98.$$

4.10 Equation 4.61 reminds you how to compute matrix elements in the *I*-*II* basis. Given

$$|I\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$
$$|II\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

we find for instance that

$$\begin{aligned}\langle I|\hat{H}|II\rangle &= \frac{1}{\sqrt{2}}(\langle 1| + \langle 2|\hat{H}|\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \\ &= \frac{1}{2}(\langle 1|\hat{H}|1\rangle - \langle 1|\hat{H}|2\rangle + \langle 2|\hat{H}|1\rangle - \langle 2|\hat{H}|2\rangle) \\ &= \frac{1}{2}\big((E_0 + \mu_e|\vec{E}_0|\cos\omega t) - (-A) + (-A) - (E_0 - \mu_e|\vec{E}_0|\cos\omega t)\big) \\ &= \mu_e|\vec{E}_0|\cos\omega t.\end{aligned}$$

For the matrix elements in the 1-2 basis I used equation 4.57, but with time-dependent electric field.

If you trace back the steps that led to Rabi's formula, you find that $\hbar\omega_0$ appears as the difference between the matrix elements H_{11} and H_{22}. For the ammonia molecule with Hamiltonian 4.61 that corresponds to $E_0 + A - (E_0 - A) = 2A$. Substituting $\hbar\omega_0 \to 2A$ and $\hbar\omega_1/2 \to \mu_e|\vec{E}_0|$

into 4.45 gives the probability of transition to the lower energy state, starting in the higher, as function of time:

$$Prob \to \frac{(\mu_e|\vec{E}_0|)^2}{(2A-\hbar\omega)^2+(\mu_e|\vec{E}_0|)^2}\sin^2\frac{\sqrt{(2A-\hbar\omega)^2+(\mu_e|\vec{E}_0|)^2}}{2\hbar}t.$$

4.12 From solutions to Ch3 homework, we have expressions for the $\hat{S}_x$ eigenstates in the S_z basis:

$$|1,1\rangle_x \to \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix} \to \frac{1}{2}|1,1\rangle_z + \frac{\sqrt{2}}{2}|1,0\rangle_z + \frac{1}{2}|1,-1\rangle_z$$
$$|1,0\rangle_x \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix} \to \frac{1}{\sqrt{2}}|1,1\rangle_z - \frac{1}{\sqrt{2}}|1,-1\rangle_z$$
$$|1,-1\rangle_x \to \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix} \to \frac{1}{2}|1,1\rangle_z - \frac{\sqrt{2}}{2}|1,0\rangle_z + \frac{1}{2}|1,-1\rangle_z$$

.

We invert these relationships to find the state $|\psi(0)\rangle = |1,1\rangle_z$ in the S_x basis,

$$|1,1\rangle_z = \frac{1}{2}|1,1\rangle_x + \frac{\sqrt{2}}{2}|1,0\rangle_x + \frac{1}{2}|1,-1\rangle_x.$$

To take the amplitude we will also eventually need

$$|1,-1\rangle_z = \frac{1}{2}|1,1\rangle_x - \frac{\sqrt{2}}{2}|1,0\rangle_x + \frac{1}{2}|1,-1\rangle_x.$$

As the S_x eigenstates are eigenstates of the Hamiltonian,

$$\hat{H}|1,1\rangle_x = \hbar\omega_0|1,1\rangle_x$$
$$\hat{H}|1,0\rangle_x = 0|1,0\rangle_x$$
$$\hat{H}|1,-1\rangle_x = -\hbar\omega_0|1,-1\rangle_x,$$

the state at time t is

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\left(\frac{1}{2}|1,1\rangle_x + \frac{\sqrt{2}}{2}|1,0\rangle_x + \frac{1}{2}|1,-1\rangle_x\right)$$
$$= \frac{1}{2}e^{-i\omega_0 t}|1,1\rangle_x + \frac{\sqrt{2}}{2}|1,0\rangle_x + \frac{1}{2}e^{+i\omega_0 t}|1,-1\rangle_x.$$

The requested amplitude is

$$
\begin{aligned}
{}_z\langle 1,-1|\psi(t)\rangle &= \left(\frac{1}{2}{}_x\langle 1,1| - \frac{\sqrt{2}}{2}{}_x\langle 1,0| + \frac{1}{2}{}_x\langle 1,-1|\right)\left(\frac{1}{2}e^{-i\omega_0 t}|1,1\rangle_x + \frac{\sqrt{2}}{2}|1,0\rangle_x + \frac{1}{2}e^{+i\omega_0 t}|1,-1\rangle_x\right) \\
&= \frac{1}{4}e^{-i\omega_0 t} - \frac{1}{2} + \frac{1}{4}e^{+i\omega_0 t},
\end{aligned}
$$

giving probability

$$
|{}_z\langle 1,-1|\psi(t)\rangle|^2 = \left(\frac{1}{2} - \frac{1}{2}\cos\omega_0 t\right)^2 = \sin^4\frac{\omega_0 t}{2}.
$$

Cf. Townsend eq. 4.27 for the comparable formula in the spin-1/2 case, where the power on the sine function is two instead of four (and, incidentally, the roles of x and z axes are reversed).

4.13 Here again we have a time-independent Hamiltonian, and the first task is to find its eigenvectors. You can solve

$$
\det\begin{pmatrix} E_0 - E & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & E_0 - E \end{pmatrix} = 0
$$

for eigenvalues E and then go back to solve for the eigenvectors. In taking the determinant it is easiest to make the minor expansion along the second row, so that

$$
\det \to (E_1 - E)((E_0 - E)^2 - A^2) = 0
$$

is already partially factored. (The principle here is to take the minor expansion along the row or column with the most zeros.)

Hopefully by now you are getting the hang of the process and can start to "eyeball" eigenvectors for symmetric matrices like this one. For instance, I see that the vector

$$
|2\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
$$

is an eigenvector, because the first and third rows are zero when acting on it and the middle row only has an element on the diagonal:

$$
\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ E_1 \\ 0 \end{pmatrix}.
$$

The eigenvalue is E_1. Immediately then we can solve part (a) of the problem, for which $|\psi(0)\rangle = |2\rangle$:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|2\rangle = e^{-iE_1 t/\hbar}|2\rangle.$$

State $|2\rangle$ is a stationary state.

To find the other eigenvectors note that the first and third rows mix the first and third slots of the column vector, and the matrix is symmetric under its transpose. If we try the symmetric and antisymmetric combinations

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\\pm 1\end{pmatrix}.$$

we will see that they are infact the eigenvectors:

$$\begin{pmatrix}E_0 & 0 & A\\ 0 & E_1 & 0\\ A & 0 & E_0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}E_0+A\\0\\A+E_0\end{pmatrix} = (E_0+A)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}E_0 & 0 & A\\ 0 & E_1 & 0\\ A & 0 & E_0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}E_0-A\\0\\A-E_0\end{pmatrix} = (E_0-A)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

The eigenvalues are $E_0 \pm A$. In part (b) we are given

$$|\psi(0)\rangle = |3\rangle \rightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

We expand

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right).$$

Then

$$\begin{aligned}|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|3\rangle &\rightarrow \left(e^{-i(E_0+A)t/\hbar}\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - e^{-i(E_0-A)t/\hbar}\frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right)\\ &= e^{-iE_0 t}\begin{pmatrix}-i\sin(At/\hbar)\\0\\\cos(At/\hbar)\end{pmatrix}.\end{aligned}$$

The amplitude sloshes back and forth between $|3\rangle$ and $|1\rangle$ as time evolves, with frequency proportional to the mixing term A in the Hamiltonian.

4.14

$$H = \begin{pmatrix}0 & -iE_0\\ iE_0 & 0\end{pmatrix}.$$

(a) Again by symmetry of the Hamiltonian we might try the symmetric and antisymmetric combinations

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}.$$

However, this time it will not work out (quite):

$$\begin{pmatrix}0 & -iE_0\\ iE_0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-iE_0\\ iE_0\end{pmatrix} \neq E\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$$

for any E. Let's fall back on the procedure and see what went wrong:

$$det\begin{pmatrix}-E & -iE_0\\ iE_0 & -E\end{pmatrix} = E^2 - E_0^2 = 0 \quad \leftrightarrow \quad E = \pm E_0.$$

Then we pick, say $E = E_0$, and solve for unknown coefficients a, b:

$$\begin{aligned}\begin{pmatrix}0 & -iE_0\\ iE_0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} &= E_0\begin{pmatrix}a\\ b\end{pmatrix}\\ \begin{pmatrix}-iE_0 b\\ iE_0 a\end{pmatrix} &= E_0\begin{pmatrix}a\\ b\end{pmatrix}\\ &\rightarrow b = ia\end{aligned}$$

Ah, so it we forgot the is. The actual combinations we want are

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix},$$

as you can check. The eigenvalues are $\pm E_0$. Since we started working in the $|X\rangle$-$|Y\rangle$ polarization basis, these eigenvectors are the circular polarizations $|R\rangle$ and $|L\rangle$.

(b) The initial polarization

$$|X\rangle \rightarrow \begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right),$$

and so

$$\begin{aligned}|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|X\rangle &\rightarrow \frac{1}{\sqrt{2}}\left(e^{-iE_0 t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + e^{+iE_0 t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right)\\ &= \begin{pmatrix}\cos E_0 t/\hbar\\ \sin E_0 t/\hbar\end{pmatrix}.\end{aligned}$$

The polarization is always real and therefore linear. It rotates in the X-Y-plane, around the optic axis.

4.15 To express the initial state $|\frac{3}{2},\frac{3}{2}\rangle_z$ in terms of the eigenstates of $\hat{S}_x$, you can apply the identity operator as a sum over $\hat{S}_x$ eigenstates:

$$\begin{aligned}|\frac{3}{2},\frac{3}{2}\rangle_z =&\hat{I}|\frac{3}{2},\frac{3}{2}\rangle_z = \sum_m |\frac{3}{2},m\rangle_{x\,x}\langle\frac{3}{2},m|\frac{3}{2},\frac{3}{2}\rangle_z \\ =&|\frac{3}{2},\frac{3}{2}\rangle_{x\,x}\langle\frac{3}{2},\frac{3}{2}|\frac{3}{2},\frac{3}{2}\rangle_z + |\frac{3}{2},\frac{1}{2}\rangle_{x\,x}\langle\frac{3}{2},\frac{1}{2}|\frac{3}{2},\frac{3}{2}\rangle_z \\ &+ |\frac{3}{2},-\frac{1}{2}\rangle_{x\,x}\langle\frac{3}{2},-\frac{1}{2}|\frac{3}{2},\frac{3}{2}\rangle_z + |\frac{3}{2},-\frac{3}{2}\rangle_{x\,x}\langle\frac{3}{2},-\frac{3}{2}|\frac{3}{2},\frac{3}{2}\rangle_z\end{aligned}$$

Use the expressions in problem 3.23 to find

$$|\frac{3}{2},\frac{3}{2}\rangle_z = \frac{1}{2\sqrt{2}}\left(|\frac{3}{2},\frac{3}{2}\rangle_x + \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle_x + \sqrt{3}|\frac{3}{2},-\frac{1}{2}\rangle_x + |\frac{3}{2},-\frac{3}{2}\rangle_x\right).$$

The energies are $\hbar\omega_0 m$, for $m = \frac{3}{2},\ \frac{1}{2},\ -\frac{1}{2},\ -\frac{3}{2}$.

$$\begin{aligned}|\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}|\frac{3}{2},\frac{3}{2}\rangle_z \\ &= \frac{1}{2\sqrt{2}}\left(e^{-3i\omega_0 t/2}|\frac{3}{2},\frac{3}{2}\rangle_x + \sqrt{3}e^{-i\omega_0 t/2}|\frac{3}{2},\frac{1}{2}\rangle_x + \sqrt{3}e^{+i\omega_0 t/2}|\frac{3}{2},-\frac{1}{2}\rangle_x + e^{+3i\omega_0 t/2}|\frac{3}{2},-\frac{3}{2}\rangle_x\right).\end{aligned}$$

Again using the expressions in 3.23 to take inner products between S_x and S_z eigenstates, the amplitude is

$$\begin{aligned}{}_z\langle\frac{3}{2},-\frac{3}{2}|\psi(t)\rangle &= \frac{1}{8}\left(e^{-3i\omega_0 t/2} - 3e^{-i\omega_0 t/2} + 3e^{+i\omega_0 t/2} - e^{+3i\omega_0 t/2}\right) \\ &= \frac{i}{4}\left(3\sin\omega_0 t/2 - \sin 3\omega_0 t/2\right).\end{aligned}$$

The probability is

$$\left|{}_z\langle\frac{3}{2},-\frac{3}{2}|\psi(t)\rangle\right|^2 = \frac{1}{16}\left(3\sin\omega_0 t/2 - \sin 3\omega_0 t/2\right)^2.$$

The motion thus described is a precession around the magnetic field along the x axis. At $t = \pi/\omega_0$,

$$\left|{}_z\langle\frac{3}{2},-\frac{3}{2}|\psi(t)\rangle\right|^2 = \frac{1}{16}\left(3\sin\pi/2 - \sin 3\pi/2\right)^2 = 1.$$

This is the half period of the motion, when the spin has flipped from maximally up along z ($S_z \to 3\hbar/2$) to maximally down ($S_z \to -3\hbar/2$).

4.16 Energy of a photon is related to the wavelength by $E = hc/\lambda$. A small *deviation* in energy is related to a small deviation in λ by

$$dE = -\frac{hc}{\lambda^2}d\lambda.$$

The time-energy uncertainty relation

$$\Delta E \Delta t \gtrsim \hbar/2$$

allows us to estimate $\Delta E \sim \hbar/\tau$ in terms of the lifetime τ. (We drop the two in this order-of-magnitude reasoning.) Putting these expressions together,

$$\Delta\lambda \sim \frac{\hbar}{\tau}\frac{\lambda^2}{hc} = \frac{\lambda^2}{2\pi\tau c} = \frac{(1.2 \cdot 10^{-5}\,\mathrm{cm})^2}{2\pi(1.6 \cdot 10^{-9}\,\mathrm{s})(3 \cdot 10^{10}\,\mathrm{cm/s})} \sim 5 \cdot 10^{-13}\,\mathrm{cm}.$$

In angstroms, the wavelength $\lambda = 1200\,\mathring{A}$, with linewidth $\Delta\lambda \sim 5 \cdot 10^{-5}\,\mathring{A}$.