Phys 125
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Solutions to problems from Chapters 7 and 9.

7.9 Use expressions for $\hat{x}$ and $\hat{p}_x$ in terms of raising and lowering operators. We are given

$$|\psi(t)\rangle = e^{-i(n+1/2)\omega t}\left(c_n|n\rangle + c_{n+1}e^{-i\omega t}|n+1\rangle\right).$$

The overall phase drops out of the expectation values. So compute, e.g.,

$$\begin{aligned}\langle p_x\rangle &= \left(\langle n|c_n^* + \langle n+1|c_{n+1}^* e^{+i\omega t}\right)\left(-i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger)\right)\left(c_n|n\rangle + c_{n+1}e^{-i\omega t}|n+1\rangle\right)\\ &= -i\sqrt{\frac{m\omega\hbar}{2}}\left(c_n^* c_{n+1}e^{-i\omega t}\langle n|\hat{a}|n+1\rangle - c_{n+1}^* e^{+i\omega t}c_n\langle n+1|\hat{a}^\dagger|n\rangle\right).\end{aligned} \tag{1}$$

The terms dropped are zero. Now, $\langle n|\hat{a}|n+1\rangle = \sqrt{n+1}\langle n|n\rangle = \sqrt{n+1}$, and similarly, $\langle n+1|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\langle n+1|n+1\rangle = \sqrt{n+1}$. Furthermore, let us call $c_n c_{n+1}^* = re^{i\delta}$ with r real. Substituting these expressions into eqn. (1) yields

$$\begin{aligned}\langle p_x\rangle &= -i\sqrt{\frac{m\omega\hbar}{2}}\left(c_n^* c_{n+1}e^{-i\omega t}\sqrt{n+1} - c_{n+1}^* e^{+i\omega t}c_n\sqrt{n+1}\right)\\ &= -i\sqrt{\frac{m\omega\hbar(n+1)}{2}}\left(re^{-i\delta}e^{-i\omega t} - re^{i\delta}e^{+i\omega t}\right)\\ &= -2r\sqrt{\frac{m\omega\hbar(n+1)}{2}}\sin(\omega t+\delta).\end{aligned}$$

With a similar computation you would find

$$\langle x\rangle = 2r\sqrt{\frac{\hbar(n+1)}{2m\omega}}\cos(\omega t+\delta).$$

With the identification $A = 2r\sqrt{\frac{\hbar(n+1)}{2m\omega}}$, the result follows.

7.12 Plugging into the energy eigenvalue equation 7.95 and taking derivatives, you should find that the equation is satisfied when $a = m\omega/2\hbar$ and $E = \hbar\omega/2$.

7.14 The ground state wavefunction is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}.$$

Since $\omega = \sqrt{g/L}$ for the simple pendulum, if you quadruple the length you cut ω in half. Call the new frequency $\omega' = \omega/2$ and the corresponding new ground state $|0'\rangle$, with wavefunction

$$\psi_{0'}(x) = \left(\frac{m\omega'}{\pi\hbar}\right)^{1/4} e^{-m\omega' x^2/2\hbar} = \left(\frac{m\omega}{2\pi\hbar}\right)^{1/4} e^{-m\omega x^2/4\hbar}.$$

The amplitude to be found in the new ground state is

$$\begin{aligned}
\langle 0'|0\rangle &= \int dx\, \psi^*_{0'}(x)\psi_0(x) \\
&= \int dx\, \left(\frac{m\omega}{2\pi\hbar}\right)^{1/4} e^{-m\omega x^2/4\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \\
&= \left(\frac{m\omega}{\sqrt{2}\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} dx\, e^{-3m\omega x^2/4\hbar} \\
&= \left(\frac{m\omega}{\sqrt{2}\pi\hbar}\right)^{1/2} \left(\frac{\pi}{3m\omega/4\hbar}\right)^{1/2} \\
&= \left(\frac{4}{3\sqrt{2}}\right)^{1/2}.
\end{aligned}$$

The integral is done by the basic rule for integration of a Gaussian. The probability is $4/3\sqrt{2} \approx .94$.

7.22 The coherent state $|\alpha\rangle$ is an eigenstate of the lowering operator,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

Correspondingly for the bra,

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*.$$

Therefore in the coherent state,

$$\langle E\rangle = \langle\alpha|\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|\alpha\rangle = \langle\alpha|\hbar\omega(\alpha^*\alpha + \frac{1}{2})|\alpha\rangle = \hbar\omega(|\alpha|^2 + \frac{1}{2}).$$

To compute $\langle E^2\rangle$, rewrite $\hat{H}^2$ using the commutation relation for the raising and lowering operators:

$$\begin{aligned}
\hat{H}^2 &= \left(\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})\right)^2 \\
&= \hbar^2\omega^2(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} + \frac{1}{4}) \\
&= \hbar^2\omega^2(\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger[\hat{a},\hat{a}^\dagger]\hat{a} + \hat{a}^\dagger\hat{a} + \frac{1}{4}) \\
&= \hbar^2\omega^2(\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + \frac{1}{4}).
\end{aligned}$$

Then,

$$\begin{aligned}\langle E^2\rangle &= \hbar^2\omega^2\langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + \frac{1}{4})|\alpha\rangle \\ &= \hbar^2\omega^2((\alpha^*)^2\alpha^2 + 2\alpha^*\alpha + \frac{1}{4}) \\ &= \hbar^2\omega^2(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4})\end{aligned}$$

Finally,

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\hbar^2\omega^2(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}) - \left(\hbar\omega(|\alpha|^2 + \frac{1}{2})\right)^2} = \hbar\omega|\alpha|.$$

Problems 2, 3) The position state representations of the harmonic oscillator energy eigenfunctions can be expressed as

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right),$$

where $H_n(z)$ is the nth Hermite polynomial. The first few are:

$$\begin{aligned} H_0(z) &= 1 \\ H_1(z) &= 2z \\ H_2(z) &= 4z^2 - 2 \\ H_1(z) &= 8z^3 - 12z \\ H_1(z) &= 16z^4 - 48z^2 + 12 \end{aligned}$$

The first three reproduce the energy eigenstates given in 7.44, 7.46, and 7.47 in the book. By repeating application of the raising operator construction on p. 255 of Townsend, you should have found expressions for the $n = 3$ and $n = 4$ cases. The point with the animation is that you should see a larger amplitude oscillation with the higher energy ($n = 3,\ n = 4$) superposition than for the lower-energy ($n = 1,\ n = 0$) case. (C.f. the formula for the amplitude of oscillation in problem 7.9.)

9.3 If the Hamiltonian is independent of time (invariant under time translations), recalling Townsend eqn. 4.16,

$$\frac{d}{dt}\langle E\rangle = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{H}]|\psi\rangle + \frac{\partial\hat{H}}{\partial t} = 0.$$

9.9 Modeling CO as a rigid rotator,

$$\hat{H} = \frac{\hat{\vec{L}}^2}{2I} = \frac{\hat{\vec{L}}^2}{2\mu r^2},$$

which has eigenvalues $E_l = \frac{l(l+1)\hbar^2}{2\mu r^2}$. For the $l = 1$ to $l = 0$ transition,

$$h\nu = E_1 - E_0 = \frac{2\hbar^2}{2\mu r^2}.$$

The reduced mass is

$$\mu = \frac{m_O \cdot m_C}{m_O + m_C} = \frac{16 \cdot 12}{16 + 12} \cdot m_p \approx 6.86 m_p,$$

where $m_p \approx m_n \approx 1.67 \cdot 10^{-24}$ g is the mass of the proton or neutron. Plugging in for the given frequency gives an inter-atomic separation of $r = 1.13 \cdot 10^{-8}$ cm.

9.10 The energy levels in the HCl harmonic oscillator are evenly spaced by amount $\hbar\omega = .37\,\text{eV}$.

(a) If the transition is between nearest-neighbor levels,

$$\begin{aligned}\lambda &= hc/.37\,\text{eV} \\ &= (6.626 \cdot 10^{-27}\,\text{erg} \cdot \text{s})(3.00 \cdot 10^{10}\,\text{cm/s})(10^4\,\mu\text{m/cm})/(.37\,\text{eV} \cdot 1.602 \cdot 10^{-12}\text{erg/eV}) \\ &= 3.35\,\mu\text{m}.\end{aligned}$$

(b) The reduced mass for HCl is, taking the atomic mass for the most common isotope of Cl,

$$\mu = \frac{m_H \cdot m_{Cl}}{m_H + m_{Cl}} = \frac{1 \cdot 35}{35 + 1} \cdot m_p \approx .9722 m_p.$$

Then,

$$.37\,\text{eV} = \hbar\omega = \hbar\sqrt{k/\mu} \quad \longrightarrow \quad k = 5.17 \cdot 10^5\,\text{g/s}^2.$$

This is 517 N/m, stiff even by standards of a macroscopic steel spring.

(c) The reduced mass for ^{37}Cl is

$$\mu_{37} = \frac{1 \cdot 37}{37 + 1} \cdot m_p \approx .9737 m_p.$$

Wavelength depends on μ through

$$\lambda = \frac{hc}{\Delta E} = \frac{hc}{\hbar\sqrt{k/\mu}} = 2\pi c\sqrt{\mu/k},$$

and so the relative shift in the spectrum from the larger reduced mass is

$$\frac{\Delta\lambda}{\lambda} = \frac{2\pi c\sqrt{\mu_{37}/k} - 2\pi c\sqrt{\mu_{35}/k}}{2\pi c\sqrt{\mu_{35}/k}} = \frac{\sqrt{.9737} - \sqrt{.9722}}{\sqrt{.9722}} = .00077.$$

The spectrometer needs resolution better than eight parts in ten thousand to resolve the presence of different isotopes.

9.11 Supposing effective moment of inertia I for the rigid rotator, the diference in energy between excited and ground states is $E_l - E_0 = l(l+1)\hbar^2/2I$, and so the population of excited state l relative to the ground state goes as

$$pop(l)/pop(0) = (2l+1)e^{-l(l+1)\hbar^2/2Ik_BT}.$$

(a) For small l,

$$pop(l)/pop(0) \approx (2l+1)(1 - l(l+1)\hbar^2/2Ik_BT + O(l^2)) = 1 + l(2 - \hbar^2/2Ik_BT) + O(l^2). \quad (2)$$

Esimating $I = \mu r^2 \sim m_p(10^{-8}\,\mathrm{cm})^2 \sim 10^{-40}\,\mathrm{g}\cdot\mathrm{cm}^2$, the coefficient on the term linear in l in the expansion goes as

$$2 - \hbar^2/2Ik_BT \sim 2 - (10^{-27}\,\mathrm{erg{\cdot}s})^2/((2{\cdot}10^{-40}\,\mathrm{g{\cdot}cm^2})({\cdot}1.38{\cdot}10^{-16}\,\mathrm{erg/K})T) \sim 2 - 100\,\mathrm{K}/3T.$$

For sufficiently large T this is clearly positive, and therefore the population is initially increasing. What is sufficiently large? Let's make a more careful estimation. In section 9.7 of Townsend the effective interatomic distance for HCl is given as 1.27 Å, and using the reduced mass from the previous problem, $I = \mu r^2 = (.9722 \cdot \mathrm{m}_p)(1.27 \cdot 10^{-8}\,\mathrm{cm})^2 = 2.62 \cdot 10^{-40}\,\mathrm{g}\cdot\mathrm{s}^2$. Then,

$$2 - \hbar^2/2Ik_BT = 2 - 15.2\,\mathrm{K}/T,$$

which is positive for temperatures above 7.6 K. I needn't have worried. For large l the exponential factor dominates and decreases with increasing l.

(b) Treat l as a continuous parameter and take a derivative of eqn. (2) with respect to l to find the maximum. This gives the condition

$$2l + 1 = \sqrt{\frac{4Ik_BT}{\hbar^2}} \quad \longrightarrow \quad l \approx 2.6,$$

for $T = 293\,\mathrm{K}$. This is consistent with Fig. 9.9 insofar as the largest absorption lines correspond to transitions from the $l = 2$ and $l = 3$ states.

9.12 Terms linear in x, y, z correspond to the $l = 1$ spherical harmonics. Using

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta,$$

and referring the formulas in Townsend eqns. 9.152, we see that

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \qquad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r}.$$

Therefore we can write

$$\psi(\vec{r}) = (x+y+z)f(r) = \left(\sqrt{\frac{2\pi}{3}}(Y_{1,-1} - Y_{1,1}) + i\sqrt{\frac{2\pi}{3}}(Y_{1,-1} + Y_{1,1}) + \sqrt{\frac{4\pi}{3}}Y_{1,0}\right) rf(r)$$
$$= \left(\frac{1+i}{\sqrt{6}}Y_{1,-1} - \frac{1-i}{\sqrt{6}}Y_{1,1} + \frac{1}{\sqrt{3}}Y_{1,0}\right)\sqrt{4\pi}rf(r).$$

In the last line I factored so you could see the relative weights of the spherical harmonics in the superposition, with the sum of the corresponding probabilities adding up to one. A measurement of $\vec{L}^2$ necessarily yields the $l = 1$ eigenvalue, $1(1+1)\hbar^2$. A measurement of L_z yields $\hbar m$ for $m = -1, 0, 1$ with equal probabilities

$$\left|\frac{1+i}{\sqrt{6}}\right|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 = \left|\frac{-(1-i)}{\sqrt{6}}\right|^2 = \frac{1}{3}.$$

9.20 We have the rigid rotator Hamiltonian $\hat{H} = \hat{\vec{L}}^2/2I$, and using the work of the previous problem, we can write the initial state

$$\psi(\theta,\phi,0) = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi = \sqrt{\frac{3}{4\pi}}\frac{y}{r} = \frac{i}{\sqrt{2}}\left(Y_{1,-1} + Y_{1,1}\right).$$

The two terms in the superposition both have $l = 1$, so they have the same energy $1(1+1)\hbar^2/2I = \hbar^2/I$ and this is a stationary state:

(a)
$$\psi(\theta,\phi,t) = e^{-iHt/\hbar}\psi(\theta,\phi,0) = e^{-i\hbar t/I}\frac{i}{\sqrt{2}}\left(Y_{1,-1} + Y_{1,1}\right).$$

(b) L_z measurements yield $\pm\hbar$ with probabilities 1/2.

(c) Inserting $\hat{L}_x$ in terms of raising and lowering operators,

$$\langle L_x\rangle = e^{+i\hbar t/I}\frac{-i}{\sqrt{2}}\left(\langle 1,-1| + \langle 1,1|\right)\left(\frac{\hat{L}_+ + \hat{L}_-}{2}\right)e^{-i\hbar t/I}\frac{i}{\sqrt{2}}\left(|1,-1\rangle + |1,-1\rangle\right) = 0,$$

because the raising/lowering operators either annihilate the ket vectors or take them to $|1,0\rangle$, which is orthogonal to the bra vectors $\langle 1,\pm 1|$.

(d) Use the results of problem 3.15 to take inner products of $|\psi(t)\rangle$ with the various states $|1,m\rangle_x$. Find $prob(L_x \to 0) = 0$ and $prob(L_x \to \pm\hbar) = 1/2$.

9.21 As for the previous problem,

$$\psi(\theta,\phi,0) = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi = \frac{i}{\sqrt{2}}\left(Y_{1,-1} + Y_{1,1}\right),$$

which gives an expansion in eigenstates of the Hamiltonian, but now, due to the external magnetic field, each term in the superposition has distinct energy. This causes rotation in the $x - y$ plane, i.e in angle ϕ. Find:

$$\psi(\theta, \phi, t) = e^{-i\hbar t/I} \sqrt{\frac{3}{4\pi}} \sin\theta \sin(\phi - \omega_0 t)$$

and

$$\langle L_x \rangle = 0.$$