Phys 125
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Solutions to Problem 6.1(c).

For Hamiltonian
$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x}),$$
show that
$$\frac{d\langle p_x \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle. \tag{1}$$

When the spread $\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ in the wavefunction is small compared to the scale over which the potential varies appreciably, then
$$\left\langle -\frac{\partial V}{\partial x} \right\rangle \approx - \left. \frac{\partial V}{\partial x} \right|_{x = \langle x \rangle},$$
and (1) can be read as saying that the expectation values of momentum and position satisfy Newton's second law. However, when the uncertainty in x is relatively large,
$$\left\langle -\frac{\partial V}{\partial x} \right\rangle$$
encodes corrections to the classical expression for force. Townsend p. 200 discusses this point in more detail.

To prove (1), use
$$\frac{d\langle p_x \rangle}{dt} = \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{p}_x] | \psi \rangle.$$
The momentum operator commutes with itself, so we need to compute
$$\langle \psi | [V(\hat{x}), \hat{p}_x] | \psi \rangle.$$

The subtlety I wanted to explain is as follows. The momentum operator can be written at a derivative *in the x-representation.* So insert the identity operator to write
$$\langle \psi | [V(\hat{x}), \hat{p}_x] | \psi \rangle = \int dx \langle \psi | x \rangle \langle x | [V(\hat{x}), \hat{p}_x] | \psi \rangle, \tag{2}$$

and then focus on the matrix element $\langle x|[V(\hat{x}),\hat{p}_x]|\psi\rangle$. To wit,

$$\begin{aligned}
\langle x|[V(\hat{x}),\hat{p}_x]|\psi\rangle &= [V(x),-i\hbar\frac{\partial}{\partial x}]\psi(x) \\
&= -i\hbar\left(V(x)\frac{\partial}{\partial x}\psi(x)-\frac{\partial}{\partial x}(V(x)\psi(x))\right) \\
&= -i\hbar\left(V(x)\frac{\partial}{\partial x}\psi(x)-\left(\psi(x)\frac{\partial V(x)}{\partial x}+V(x)\frac{\partial\psi(x)}{\partial x}\right)\right) \\
&= +i\hbar\psi(x)\frac{\partial V(x)}{\partial x}.
\end{aligned}$$

The point is that the derivative operator acts on both $\psi(x)$ and $V(x)$, and in the second term in the commutator you have to apply the product rule. Always write in a wavefunction $\psi(x)$ when working out commutators involving $\hat{p}_x$. Reinserting the matrix element into (2),

$$\begin{aligned}
\int dx\langle\psi|x\rangle\langle x|[V(\hat{x}),\hat{p}_x]|\psi\rangle &= \int dx\,\psi^*(x)\left(i\hbar\psi(x)\frac{\partial V(x)}{\partial x}\right) \\
&= i\hbar\langle\psi|\frac{\partial V(x)}{\partial x}|\psi\rangle.
\end{aligned}$$

The result follows.