Phys 125
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Solutions to the short midterm.

1. A spin-1/2 particle is in state

$$|\psi\rangle = N\,(3|\boldsymbol{z}+\rangle + i|\boldsymbol{z}-\rangle) \to N\begin{pmatrix}3\\ i\end{pmatrix}.$$

The matrix representation of $|\psi\rangle$ and everything below is in the basis of $\hat{S}_z$ eigenstates.

(a) Normalize:

$$1 = \langle\psi|\psi\rangle = N^*\begin{pmatrix}3 & -i\end{pmatrix} N\begin{pmatrix}3\\ i\end{pmatrix} = 10|N|^2.$$

We can choose the overall phase so that N is real,

$$N = \frac{1}{\sqrt{10}}.$$

(b) First let's do the relevant amplitudes.

$$\begin{aligned}\langle x+|\psi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix}3\\ i\end{pmatrix}\\ &= \frac{1}{\sqrt{20}}(3+i);\\ \langle y+|\psi\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix}3\\ i\end{pmatrix}\\ &= \frac{1}{\sqrt{20}}(3+1) = \frac{2}{\sqrt{5}}.\end{aligned}$$

The $-i$ in the $|y+\rangle$ amplitude appears from complex conjugating to get the bra vector. The probabilities are then

$$\begin{aligned}Prob(S_x \to +\frac{\hbar}{2}) &= \left|\frac{1}{\sqrt{20}}(3+i)\right|^2 = \frac{(3+i)(3-i)}{20} = \frac{1}{2};\\ Prob(S_y \to +\frac{\hbar}{2}) &= \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5}.\end{aligned}$$

At this point, it's good to think through the result. The state

$$|\psi\rangle \to \frac{1}{\sqrt{10}}\begin{pmatrix}3\\ i\end{pmatrix}$$

is in the $y - z$ plane, which we can see from the fact that there is a relative phase of $i = e^{i\pi/2}$ between the z-spin-up and z-spin-down components of the superposition. So it makes sense that we have no information about S_x: measurement along the x axis is a 50-50 proposition. On the other hand, this state is mostly (4/5) up along the y axis.

(c) The matrix representations of the spin operators are

$$\hat{S}_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y \to \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z \to \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(I have given S_z for completeness.) We compute:

$$\begin{aligned}
\langle\psi|\hat{S}_x|\psi\rangle &= \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ i \end{pmatrix} \\
&= \frac{\hbar}{20}\begin{pmatrix} 3 & -i \end{pmatrix}\begin{pmatrix} i \\ 3 \end{pmatrix} \\
&= \frac{\hbar}{20}(3i - 3i) = 0. \\
\langle\psi|\hat{S}_y|\psi\rangle &= \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & -i \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ i \end{pmatrix} \\
&= \frac{\hbar}{20}\begin{pmatrix} 3 & -i \end{pmatrix}\begin{pmatrix} 1 \\ 3i \end{pmatrix} \\
&= \frac{\hbar}{20}(3 + 3) = \frac{3\hbar}{10}.
\end{aligned}$$

This meshes with our previous analysis. Along x the spin is 50/50 up or down, and it is more up than down along y. We could have equivalently computed this result as

$$\begin{aligned}
\langle S_x\rangle &= \frac{\hbar}{2}\cdot Prob(S_x \to +\frac{\hbar}{2}) - \frac{\hbar}{2}\cdot Prob(S_x \to -\frac{\hbar}{2}) \\
&= \frac{\hbar}{2}\cdot\frac{1}{2} - \frac{\hbar}{2}\cdot\frac{1}{2} = 0; \\
\langle S_y\rangle &= \frac{\hbar}{2}\cdot Prob(S_y \to +\frac{\hbar}{2}) - \frac{\hbar}{2}\cdot Prob(S_y \to -\frac{\hbar}{2}) \\
&= \frac{\hbar}{2}\cdot\frac{4}{5} - \frac{\hbar}{2}\cdot\frac{1}{5} = \frac{3\hbar}{10}.
\end{aligned}$$

(d)

$$\hat{R}(\theta\hat{z}) = e^{-i\theta\hat{S}_z/\hbar}.$$

We are working the in basis of $\hat{S}_z$ eigenstates, for which

$$\begin{aligned}
e^{-i\theta\hat{S}_z/\hbar}|z+\rangle &= e^{-i\theta(\frac{\hbar}{2})/\hbar}|z+\rangle = e^{-i\theta/2}|z+\rangle \\
e^{-i\theta\hat{S}_z/\hbar}|z-\rangle &= e^{-i\theta(-\frac{\hbar}{2})/\hbar}|z-\rangle = e^{+i\theta/2}|z-\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{R}(\frac{\pi}{2}\hat{z})|\psi\rangle &= e^{-i\frac{\pi}{2}\hat{S}_z/\hbar}\frac{1}{\sqrt{10}}\left(3|\boldsymbol{z}+\rangle + i|\boldsymbol{z}-\rangle\right) \\
&= \frac{1}{\sqrt{10}}\left(3e^{-i\pi/4}|\boldsymbol{z}+\rangle + ie^{+i\pi/4}|\boldsymbol{z}-\rangle\right) \\
&= \frac{e^{-i\pi/4}}{\sqrt{10}}\left(3|\boldsymbol{z}+\rangle - |\boldsymbol{z}-\rangle\right).
\end{aligned}$$

I factored out the overall $e^{-i\pi/4}$ because it helps you see that this state lies now in the $x-z$ plane, more toward $|x-\rangle$ than $|x+\rangle$.

(e) Before computing anything, you can reason your way to the answer. We have rotated the spin, which started in the $y-z$ plane, by $\pi/2$ around the z axis. It is now in the $x-z$ plane, weighted toward $|x-\rangle$ just as much as it had been weighted toward $|y+\rangle$. We should have

$$\begin{aligned}
\langle S_x\rangle &= -\frac{3\hbar}{10}; \\
\langle S_y\rangle &= 0.
\end{aligned}$$

I probably would compute $\langle S_x\rangle$ to be sure:

$$\begin{aligned}
\langle\psi|\hat{S}_x|\psi\rangle &= \frac{e^{i\pi/4}}{\sqrt{10}}\begin{pmatrix}3 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{e^{-i\pi/4}}{\sqrt{10}}\begin{pmatrix}3\\ -1\end{pmatrix} \\
&= \frac{\hbar}{20}\begin{pmatrix}3 & -1\end{pmatrix}\begin{pmatrix}-1\\ 3\end{pmatrix} \\
&= \frac{\hbar}{20}(-3-3) = -\frac{3\hbar}{10}.
\end{aligned}$$

2. Suppose a spin-1/2 particle is in the state $|z+\rangle$. When we put this particle through an SG$\hat{x}$ Stern-Gerlach apparatus, we observe that half the particles bend up, half bend down, which is to say, we measure angular momentum $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ along the x axis. Even when the spin is "spin-up along the z axis," we measure non-zero angular momentum along x; in no sense can we say the angular momentum is strictly along z. You could make the same argument for any axis you choose.

The fact that you measure non-zero angular momentum along axes other than z is also expressed in the Heisenberg uncertainty relation,

$$\Delta S_x \Delta S_y \geq \frac{1}{2}\left|\langle[\hat{S}_x, \hat{S}_y]\rangle\right| = \frac{\hbar}{2}\left|\langle\hat{S}_z\rangle\right|.$$

In the state $|z+\rangle$, $\langle\hat{S}_z\rangle = \frac{\hbar}{2}$. The right-hand side of the uncertainty relation gives a bound strictly greater than zero on both ΔS_x and ΔS_y. If the uncertainties are greater than zero, then some measurements of S_x and S_y must yield values that are non-zero, and we can again claim that angular momentum is not strictly along z.