9/19 In Class Problems – Velocity 4-vectors

1. I don't think people finished this one from 9/14 in class. (If you did, please do it again now anyway.) Calculate the velocity addition formula by starting with:

$$u = \frac{dx}{dt} = \frac{\frac{dx}{dt'}}{\frac{dt}{dt'}}$$

   Use the Lorentz Transforms for x and t in terms of x' and t' to do the derivatives.

2. Given a particle moving at speed $u = \frac{4}{5}c$ in frame S, and a Frame S' moving at speed $v = \frac{3}{5}c$ relative to S, Find the speed of the particle relative to S'. (ie, find u' – you will need the other velocity "addition" rule.)

3. Griffiths defines *proper velocity*, η as $\eta = \frac{dx}{d\tau}$ where x is the position in the lab frame, and τ is the time in the particle's rest frame! (It's a very odd idea to mix frames like this, but my intent with this worksheet is to show you why it's a cool choice.) Let's say there are only those two frames. Particle frame and lab frame. So, the particle is moving at speed v relative to the lab. Time in the lab frame would be *dilated* by a factor of γ, ie, $dt = \gamma d\tau$. (Remember "moving clocks run slow." Meaning, the particle's clock is slower than ours. We in lab see a longer lifetime, say, for the particle.)

   Show that proper velocity, η, is equal to γv where v is the speed of the particle relative to the lab frame.

4. Proper velocity 4-vector: This is very often just called the velocity 4-vector, or the relativistic velocity 4-vector.

   Griffiths goes on to define $\eta^\mu = \gamma \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix}$

   (a) Show that the 4-vector $\eta = \gamma \frac{dx}{dt}$ where x is the position-time 4-vector. (You are really only checking η^0 since the others are pretty much already done by definition.)

   (b) Show that $\eta_\mu \eta^\mu$ is invariant. Comment.

5. Calculate the components of the velocity 4-vector for the particle in problem 2 (moving at speed $\frac{4}{5}c$ in the positive x-direction.)

6. Because only the numerator of $\eta = \frac{dx}{d\tau}$ transforms under the Lorentz Transformations when you change frames (proper time is invariant by definition), velocity 4-vectors transform by the Lorentz Transformations! That means, you can calculate $\eta^{\mu'} = \Lambda \eta^\mu$ where Λ is the matrix form of the Lorentz Transformations.

   (a) Write out the Λ matrix for a frame moving at speed $v = \frac{3}{5}c$ in the x dir. Recall that:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Use the above matrix to calculate $\eta^{\mu\prime} = \Lambda\eta^{\mu}$ for the velocity 4-vector you calculated in problem 5. Compare your answer to problem 2. Is it as you expect? How many "gamma"s appear in this problem (if you had to write it from scratch)? Identify them clearly.