Ahead of class on March 2, please read Thorn et al., "Observing the quantum behavior of light in an undergraduate laboratory," and then take on the following two problems:

Question 1. We derived that the first order approximation for the coefficient of the m^{th} unperturbed state in the superposition of states describing the eigenstate of the perturbed Hamiltonian (i.e. $H_0 + \lambda H'$) is

$$c_m^1(t) = -\frac{i}{\hbar}\int_0^t H'_{mn} e^{\frac{i}{\hbar}(E_m - E_n)t} dt$$

For our semi-classical analysis of the photoelectric effect, we use a classical description of the electromagnetic wave. This means the potential energy change of the electron due to its interaction with the wave is

$$H' = V(\mathbf{r}, t) = -e\mathcal{E}_0 x \cos\omega t$$

We can compute $c_m^1(t)$:

$$c_m^1(t) = -\frac{ie\mathcal{E}_0}{\hbar}\chi_{mn}\int_0^t e^{i\omega_{mn}t}\cos(\omega t)dt$$

where

$$\omega_{mn} = \frac{(E_m - E_n)}{\hbar}$$

and

$$\chi_{mn} = \langle\phi_m| x |\phi_n\rangle$$

Your task is to compute the time integral in this expression. Show that the answer is

$$D(t) = \frac{i}{2}\left(\frac{e^{i(\omega-\omega_{mn})t} - 1}{(\omega_{mn} - \omega)} - \frac{e^{i(\omega_{mn}+\omega)t} - 1}{(\omega_{mn} + \omega)}\right)$$

So for the semi-classical analysis of the photo electric effect, the amplitude for making an electron making a transition from the state n to the final state m is

$$c_m^1(t) = -\frac{ie\mathcal{E}}{\hbar}\chi_{mn}D(t)$$

The function $D(t)$ is dominated by frequencies of the electromagnetic wave such that $\omega \approx \omega_{mn}$. Explain why this is so and then take the limit that $\omega - \omega_{mn} << \omega + \omega_{mn}$ and show that in this limit

$$|D(t)|^2 = \frac{\sin^2[(\omega - \omega_{mn})t/2]}{(\omega - \omega_{mn})^2}$$

Question 2. Assume an electron transitions from a bound state $|\phi_g\rangle$ in an atom (or a large collection of atoms, such as a solid), for which we take the energy to be 0, to a final free electron state having energy $E_f = E + E_I$, where E_I is the ionization energy needed to unbind the electron from the atom. The probability of this transition is given (in the lowest order approximation) by the square of $c_f^1(t)$:

$$
\begin{aligned}
P(t) = \left|c_f^1(t)\right|^2 &= \left|\frac{ie\mathcal{E}_0}{\hbar}\chi_{fg}\right|^2 \frac{\sin^2\left[(\omega-\omega_{fg})t/2\right]}{(\omega-\omega_{fg})^2} \\
&= \left(\frac{e\mathcal{E}_0}{\hbar}\right)^2 \left|\chi_{fg}\right|^2 \left(\frac{t}{2}\right)^2 \frac{\sin^2\left[(\omega-\omega_{fg})t/2\right]}{\left[(\omega-\omega_{fg})t/2\right]^2}
\end{aligned}
$$

To get the total probability that an electron is liberated from the material due to its interaction with the incident light, we need to account for all final energies that are possible:

$$
\bar{P}(t) = \int_0^\infty g(E)P(t)dE,
$$

where $g(E)$ is the density of final states available to the electron between E and $E + dE$. The function

$$
\frac{\sin^2\left[(\omega-\omega_{fg})t/2\right]}{\left[(\omega-\omega_{fg})t/2\right]^2}
$$

is highly peaked around $\omega = \omega_{fg}$. Note, that $\omega_{fg} = (E + E_I)/\hbar$. For long times, this function becomes the Dirac delta function:

$$
\lim_{t\to\infty} \frac{\sin^2\left[(\omega-\omega_{fg})t/2\right]}{\left[(\omega-\omega_{fg})t/2\right]^2} = \frac{2\pi}{t}\delta(\omega-\omega_{fg})
$$

So, we get

$$
\begin{aligned}
\bar{P}(t) &= \tfrac{2\pi}{\hbar t}\left(e\mathcal{E}_0\right)^2 \left|\chi_{fg}(\hbar\omega - E_I)\right|^2 \left(\tfrac{t}{2}\right)^2 g(\hbar\omega - E_I) \\
&= \tfrac{\pi}{2\hbar}\left(e\mathcal{E}_0\right)^2 \left|\chi_{fg}(\hbar\omega - E_I)\right|^2 g(\hbar\omega - E_I)t
\end{aligned}
$$

You don't need to do any details calculations, but explain how the result above contains the properties of the photo electric effect:

- The ejected electrons come off almost instantaneously.
- There is a threshold frequency; below this frequency not electrons are ejected.
- The number of electrons being ejected is proportional to the intensity of the light.
- The energy of the electron is determined by the light's frequency.