Phys 140 - QoR
Boyda/Wensley

Problems due March 28:

1. In Appendix B of their paper "Interference with correlated photons: Five quantum mechanics experiments for undergraduates," Galvez et al. explain certain necessary conditions on the process of spontaneous parametric downconversion and give empirical formulas for the indices of refraction of the uniaxial BBO birefringent crystal.

   The angle θ at which the signal photon leaves the crystal is determined by refraction at the back surface of the crystal (Snell's law) and the angle θ_c which the signal photon makes relative to the incoming pump beam when generated inside the crystal. The angle θ_c is determined by the phase matching condition,

   $$n_p = n_s \cos\theta_c.$$

   In this formula, n_p is the index of refraction for the pump beam (necessarily the extraordinary index $\tilde{n}_e(\theta_m)$) and n_s is the index of refraction for the signal photon (the ordinary index in the crystal n_o). The extraordinary index depends on the wavelength of the light ($\lambda_p = 405\,\text{nm}$) and the angle θ_m the beam makes with the optical axis of the crystal, while n_o depends on the wavelength of the signal photon ($\lambda_s = 810\,\text{nm}$) alone.

   (a) Determine the angle θ_m between the direction of the pump beam $\vec{k}_p$ and the optical axis of the crystal if the signal photon is to leave the crystal at an angle $\theta = 5°$ relative to $\vec{k}_p$.

   (b) Determine θ_m if the signal photon is to exit along the direction $\vec{k}_p$ (i.e. $\theta = 0$).

2. A projection operator $\hat{P}$ satisfies $\hat{P}^2 = \hat{P}$. Which is to say, repeating a projection has no additional effect. For a unit vector $|u\rangle$ in an n-dimensional Hilbert space, show that $\hat{P}_u = |u\rangle\langle u|$ is a projection operator. Additionally, show that $\hat{P}_u$ has eigenvalues 0 and 1.[1]

3. Let $\{|e_i\rangle\}$, $i = 1 \ldots n$, be an orthonormal basis. By acting on an arbitrary state $|\psi\rangle = \sum_{j=1}^{n} c_j |e_j\rangle$ show that the identity operator can be written

   $$\hat{I} = \sum_{i=1}^{n} |e_i\rangle\langle e_i|.$$

[1] You may use the fact that for any such $|u\rangle$ there exists an orthonormal basis containing $|u\rangle$.

4. The exponential of an operator $\hat{A}$ is defined by its Taylor expansion:

$$e^{\hat{A}} = 1 + \hat{A} + \frac{\hat{A}^2}{2!} + \frac{\hat{A}^3}{3!} + \ldots.$$

The angular moment operator $\hat{l}_z$ for a photon traveling in the z direction can expressed in the horizontal-vertical polarization basis as the matrix

$$l_z = \begin{pmatrix} 0 & i\hbar \\ -i\hbar & 0 \end{pmatrix}.$$

Show that $\hat{R}(\theta) = e^{i\theta\hat{l}_z/\hbar}$ rotates a linear polarization by angle θ. We say that $\hat{l}_z$ is the *generator* of rotations about the z axis.

5. The commutator of two operators $\hat{A}$ and $\hat{B}$ is

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}.$$

Two operators are said to *commute* if their commutator is zero, i.e., if $\hat{A}\hat{B} = \hat{B}\hat{A}$.

In an n-dimensional Hilbert space, suppose that $\hat{A}$ and $\hat{B}$ each have n distinct eigenvalues. Show that $\hat{A}$ and $\hat{B}$ commute if and only if there exists a basis consisting of vectors which are simultaneously eigenvectors of both operators.[2]

6. The next two problems refer to the spin angular momentum of an electron. In the basis spin-up / spin-down along the z axis, the spin matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Find normalized eigenvectors and eigenvalues of each of the S_i.
   (b) Show that $[S_x, S_y] = i\hbar S_z$ and analogous results for $[S_y, S_z]$ and $[S_z, S_x]$.

7. In a constant magnetic field $\vec{B} = B_z\hat{z}$ has Hamiltonian

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = \frac{geB_z}{2m}\hat{S}_z.$$

   (a) At time $t = 0$ we prepare the system with spin up along the z axis. What is the state at some later time t? What is the probability of measuring the spin along the z axis and finding spin $\hbar/2$? Of finding $-\hbar/2$?

[2] You may use the fact that an operator is wholly determined by its action on a set of basis vectors.

(b) Now instead at $t = 0$ we prepare the system in the initial state

$$|\psi(0)\rangle = \frac{\sqrt{3}}{2}|z+\rangle + \frac{1}{2}|z-\rangle.$$

What is the state $|\psi(t)\rangle$ at some later time t? As a function of t, compute the probabilities for finding the electron to be spin-up or spin-down along the z axis, spin-up or spin-down along the y axis, spin-up or spin-down along the x axis, six separate probabilities. In words, how would you describe the time-evolution of the electron?