Short answer

1. Physics is *local* if any physical process affects only its immediate neighborhood. In the strictest sense, interactions are local if they happen at a spacetime point. Any influence beyond the immediate neighborhood propagates at speed less than or equal to the speed of light.

2. EPR propose, as one possible definition, that a physical quantity be deemed real if it can be measured without disturbing the relevant physical system. For instance, if two electron spins are in an entangled singlet state, by measuring the first in the $|x\pm\rangle$ basis one can determine the x spin of the other. And similarly, by measuring the first in the $|z\pm\rangle$ basis, one could determine the z spin of the other. Since neither operation disturbs the second spin, the x and z spins must both be elements of physicality reality according to EPR.

   That is sufficient answer for the problem. However, the word *disturb* merits further explication, because in their operational example EPR presume locality, that a particle cannot be disturbed by a measurement performed far away. In quantum mechanics, an entangled wavefunction is collapsed by such measurements, and this collapse has observable consequences (Bell correlations). Taking the EPR definition while giving full credence to the quantum wavefunction, one would conclude that the elements of reality are precisely those described by quantum mechanics, and that they depend on experimental circumstance. Today researchers prefer to speak of *counterfactual definiteness* – whether the quantities one could measure in principle, but choses not to (which are thereby *counter-factual*), e.g. z spins in an x-eigenstate, are nonetheless *definitely* determined (in the sense that measurement outcomes could in principle be known with certainty) in some underlying physics.

3. Hidden variables are physical quantities, or perhaps their mathematical representations, hypothesized to determine the elements of physical reality that are not represented in our current physical theory (quantum mechanics).

4. It is possible to create two photons – two particles of light – perhaps in the double decay of an atom – and send them off in two different directions, kilometers apart if you like, in such a way that they still behave as if part of one quantum entity. Even though what happens to each independently it totally random, what happens to one is instantaneously reflected in what happens to the other.

   Say in each place there is a mechanism to detect the polarization (or spin) of the photon – depending on polarization, the photon either passes or does not pass through the detector. If you make identical measurements, you can arrange that the two outcomes are identical. Both photons pass, or both don't. Nothing is so remarkable in that. You could imagine, for instance, that you promised your grandkids identical graduation presents. Trusting you to keep your word, the one in Wisconsin, on opening her envelope and seeing a dollar bill, would immediately know that a dollar bill just arrived at her cousin's in Florida.

Now suppose you measure your two photons with different detectors settings in the two locations – one detector rotated slightly relative to the other. The outcomes will not definitely be identical, there being randomness in the process, but they will still be more likely to be the same than if they were truly disconnected, independent entities, of which all properties were determined at the moment of leaving the original atom. The analogy with the dollar bills breaks down here, but there is a theorem which states that the quantum photons will tend more often to identical outcomes than independent, classical entities. Experimenters have checked, and found these quantum correlations in the laboratory. Even separated by kilometers, the photons seem somehow to know what is happening to the other.

There is no underlying physical reality where all measurable quantities are definitely determined, at least not one that respects local cause and effect. Instead, there is quantum mechanics.

Problem 1
Use the spin 1/2 expectation value:

$$E(a,b) = -\cos\phi,$$

where ϕ is the angle between polarizer settings $\vec{a}$ and $\vec{b}$, and check the inequality

$$1 + E(b,c) \geq |E(a,b) - E(a,c)|$$

Using $\phi = 60^\circ$, first compute the LHS:

$$1 + E(b,c) = 1 - \cos\phi = 1 - \cos 60^\circ = 1 - \frac{1}{2} = \frac{1}{2}.$$

Now compute the RHS:

$$|E(a,b) - E(a,c)| = |-\cos\phi + \cos 2\phi| = |-\cos 60^\circ + \cos 120^\circ| = \left|-\frac{1}{2} - \frac{1}{2}\right| = 1.$$

Since,

$$\frac{1}{2} \not\geq 1,$$

The inequality is violated.

Problem 2
Use the HVT result

$$E(a,b) = \frac{2\phi}{\pi} - 1$$

Compute the LHS of the inequality (see Problem 1):

$$1 + \frac{2\left(\frac{\pi}{3}\right)}{\pi} - 1 = \frac{2}{3}.$$

Now compute the RHS:

$$\left|\frac{2\phi}{\pi} - 1 - \left(\frac{2\,(2\phi)}{\pi} - 1\right)\right| = \left|\frac{2(\frac{\pi}{3})}{\pi} - 1 - \left(\frac{2\left(\frac{2\pi}{3}\right)}{\pi} - 1\right)\right| = \frac{2}{3}$$

Since,

$$\frac{2}{3} \geq \frac{2}{3},$$

the inequality is satisfied.

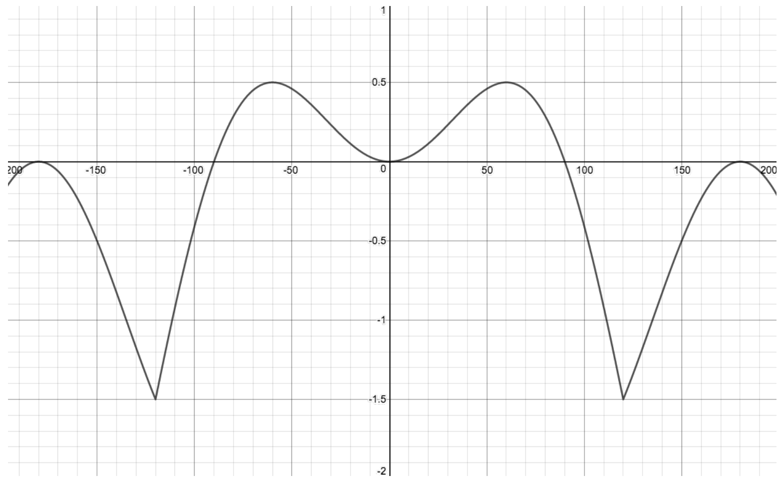
Problem 3
I want to check the following inequality for $\phi \in (-180^\circ, +180^\circ]$:

$$1 - \cos\phi \geq |-\cos\phi + \cos 2\phi|$$

Rearranging, this is equivalent to checking

$$0 \geq \cos\phi + |-\cos\phi + \cos 2\phi| - 1.$$

I plot the RHS of this inequality between -180° and 180° and look for points that will be ≤ 0:



From the graph, there is a large range of angles that do not satisfy the inequality. Excluding $\phi = 0$, all angles in the range $-90^\circ < \phi < 90^\circ$ violate the inequality.

Problem 4

Using the product state

$$|\psi\rangle = |\psi\rangle_A |\psi\rangle_B \,,$$

and its adjoint

$$\langle\psi| = \langle\psi|_B \langle\psi|_A \,,$$

I compute the expectation value, noting that the operators labeled A (B) only operate on the states labeled A (B):

$$E(a,b) = \langle\psi|_B \langle\psi|_A (\vec{\sigma}_A \cdot \hat{a})(\vec{\sigma}_B \cdot \hat{b}) |\psi\rangle_A |\psi\rangle_B = \langle\psi|_A (\vec{\sigma}_A \cdot \hat{a}) \langle\psi|_B (\vec{\sigma}_B \cdot \hat{b}) |\psi\rangle_A |\psi\rangle_B$$

$$\longrightarrow E(a,b) = \langle\psi|_A (\vec{\sigma}_A \cdot \hat{a}) |\psi\rangle_A \langle\psi|_B (\vec{\sigma}_B \cdot \hat{b}) |\psi\rangle_B$$

This is just the product of the expectation values for each subspace.

Problem 5

Using the state

$$|\psi\rangle = |z+\rangle_A |z-\rangle_B \,,$$

to compute $E(a,b)$ use the result from Problem 4:

$$E(a,b) = \langle z+| \sigma_x a_x + \sigma_y a_y + \sigma_z a_z |z+\rangle \langle z-| \sigma_x b_x + \sigma_y b_y + \sigma_z b_z |z-\rangle \tag{1}$$

I use the following results to compute the expectation values:

$$\begin{aligned} \sigma_z |z+\rangle &= + |z+\rangle \\ \sigma_z |z-\rangle &= - |z-\rangle \end{aligned}$$

For the other σ operators:

$$\sigma_y |z+\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\,|z-\rangle$$

With this result and repeating the matrix algebra for the other σ matrices and $|z\pm\rangle$ yields

$$\begin{aligned}\sigma_y \left|z+\right\rangle &= +i\left|z-\right\rangle \\ \sigma_y \left|z-\right\rangle &= -i\left|z+\right\rangle \\ \sigma_x \left|z+\right\rangle &= +\left|z-\right\rangle \\ \sigma_x \left|z-\right\rangle &= +\left|z+\right\rangle\end{aligned}$$

With these relationships, the following expectation values of the various σ matrices are found:

$$\left\langle z+\right|\sigma_y\left|z+\right\rangle = i\left\langle z+\right|\left.z-\right\rangle = 0$$

Likewise,

$$\begin{aligned}\left\langle z-\right|\sigma_y\left|z-\right\rangle &= 0 \\ \left\langle z-\right|\sigma_x\left|z-\right\rangle &= 0 \\ \left\langle z+\right|\sigma_x\left|z+\right\rangle &= 0\end{aligned}$$

Using these results in (1) yeilds

$$E(a,b) = \left\langle z+\right|\sigma_z a_z\left|z+\right\rangle\left\langle z-\right|\sigma_z b_z\left|z-\right\rangle = a_z b_z\left\langle z+\right|\sigma_z\left|z+\right\rangle\left\langle z-\right|\sigma_z\left|z-\right\rangle = a_z b_z(+1)(-1)$$

$$\longrightarrow E(a,b) = -a_z b_z \tag{2}$$

Constrast this with (2) in Problem 1, which is for an entangled state:

$$E(a,b) = -\cos\phi = -\mathbf{a}\cdot\mathbf{b} = -(a_x b_x + a_y b_y + a_z b_z)$$

For the product state only the z coefficients contribute to $E(a,b)$. This is because with the entangled state we have terms such as $\left\langle z+\right|\sigma_y\left|z-\right\rangle$, which is not zero.

Problem 6
Bell's inequality using $E(a,b)$ from the previous problem, is

$$1 + (-b_z c_z) \geq \left|-a_z b_z - (-a_z c_z)\right| \rightarrow 1 - b_z c_z \geq |a_z|\,|c_z - b_z|$$

I can rewrite this inequality as

$$|a_z|\,|c_z - b_z| + b_z c_z - 1 \leq 0 \tag{3}$$

Since $\hat{a}$, $\hat{b}$and $\hat{c}$ are unit vectors, $a_z, b_z, c_z \in (-1,1)$. The inequality can be checked by numerically checking values of a_z, b_z, c_z using nested loops. Below is a program that checks all values between $(-1,1)$ with an increment of 0.01. Note that since only the absolute value of a_z occurs, a_z can run from 0 to 1. The program found no values of the RHS of (3) that were greater than zero, which would violate the inequality.

```
/* Bell's inequality for product state  */
#include <math.h>
#include <stdio.h>

main()
{
double a,b,c,q;

q=0;
a=0;
while(a<=1.0)
{
b=-1.;
while(b<=1.0)
   {
   c=-1.;
   while(c<=1.0)
     {
     q=c*b+a*fabs(c-b)-1.;
     if (q>0)
     printf("%lf %lf %lf %lf \n",a,b,c,q);
     c=c+.01;
     }
   b=b+.01;
   }
a=a+.01;
}
}
```

PW Problem 4.
Use marginalization:

$$n(\bar{a},c) = n(\bar{a},b,c) + n(\bar{a},\bar{b},c)$$

$$n(b,\bar{c}) = n(a,b,\bar{c}) + n(\bar{a},b,\bar{c})$$

Now add,

$$\begin{aligned} n(\bar{a},c) + n(b,\bar{c}) &= n(\bar{a},b,c) + n(\bar{a},\bar{b},c) + n(a,b,\bar{c}) + n(\bar{a},b,\bar{c}) \\ &= n(\bar{a},b,c) + n(\bar{a},b,\bar{c}) + n(\bar{a},\bar{b},c) + n(a,b,\bar{c}) \\ &= n(\bar{a},b) + n(\bar{a},\bar{b},c) + n(a,b,\bar{c}) \end{aligned}$$

$$\longrightarrow n(\bar{a},c) + n(b,\bar{c}) \geq n(\bar{a},b)$$

PW Problem 5.

Compute the inner products for the two polarizer orientations relative to $|v\rangle$ and $|h\rangle$ (see lecture notes):

$$\begin{aligned}
\langle v'|\, v\rangle &= \cos\theta_1 \\
\langle h'|\, v\rangle &= -\sin\theta_1 \\
\langle v'|\, h\rangle &= \sin\theta_1 \\
\langle h'|\, h\rangle &= \cos\theta_1 \\
\langle v''|\, v\rangle &= \cos\theta_2 \\
\langle h''|\, v\rangle &= -\sin\theta_2 \\
\langle v''|\, h\rangle &= \sin\theta_2 \\
\langle h''|\, h\rangle &= \cos\theta_2
\end{aligned}$$

The input wave function to the two polarizers (one in orientation $\hat{a}$ and the other in orientation $\hat{b}$) is

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}\left(|v_a\rangle\,|v_b\rangle + |h_a\rangle\,|h_b\rangle\right)$$

The possible output states are

$$|\psi_{\text{out}}\rangle = \begin{cases} |v'_a\rangle\,|v''_b\rangle \\ |v'_a\rangle\,|h''_b\rangle \\ |h'_a\rangle\,|v''_b\rangle \\ |h'_a\rangle\,|h''_b\rangle \end{cases}$$

Compute the possible probabilities for the four outcomes. For single channel polarizers, I assume only the vertical components pass through. So, the probability that both polarizers pass a photon is

$$\begin{aligned}
P_{++} &= \left|\langle v'_a|\,\langle v''_b|\ \ \psi_{\text{in}}\rangle\right|^2 \\
&= \left|\langle v'_a|\,\langle v''_b|\,\tfrac{1}{\sqrt{2}}\left(|v_a\rangle\,|v_b\rangle + |h_a\rangle\,|h_b\rangle\right)\right|^2 \\
&= \tfrac{1}{2}\left|\langle v'_a|\, v_a\rangle\,\langle v''_a|\, v_b\rangle + \langle v'_a|\, h_a\rangle\,\langle v''_a|\, h_b\rangle\right|^2 \\
&= \tfrac{1}{2}\left|\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right|^2 \\
&= \tfrac{1}{2}\cos^2(\theta_1 - \theta_2)
\end{aligned}$$

The probability that a passes a photon and b does not is

$$\begin{aligned} P_{+-} &= |\langle v'_a| \langle h''_b| \ \ \psi_{\text{in}}\rangle|^2 \\ &= \left| \langle v'_a| \langle h''_b| \tfrac{1}{\sqrt{2}} \left(|v_a\rangle |v_b\rangle + |h_a\rangle |h_b\rangle \right) \right|^2 \\ &= \tfrac{1}{2} \left| \langle v'_a| \ v_a\rangle \langle h''_a| \ v_b\rangle + \langle v'_a| \ h_a\rangle \langle h''_a| \ h_b\rangle \right|^2 \\ &= \tfrac{1}{2} \left| \cos\theta_1(-\sin\theta_2) + \sin\theta_1 \cos\theta_2 \right|^2 \\ &= \tfrac{1}{2} \left| \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \right|^2 \\ &= \tfrac{1}{2} \sin^2(\theta_1 - \theta_2) \end{aligned}$$

Similarly,

$$P_{-+} = \left| \langle h'_a| \langle v''_b| \frac{1}{\sqrt{2}} \left(|v_a\rangle |v_b\rangle + |h_a\rangle |h_b\rangle \right) \right|^2 = \frac{1}{2} \left| -\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 \right|^2$$

$$\longrightarrow P_{-+} = \frac{1}{2} \sin^2(\theta_2 - \theta_1)$$

$$P_{--} = \left| \langle h'_a| \langle h''_b| \frac{1}{\sqrt{2}} \left(|v_a\rangle |v_b\rangle + |h_a\rangle |h_b\rangle \right) \right|^2 = \frac{1}{2} \left| (-\sin\theta_1)(-\sin\theta_2) + \cos\theta_1 \cos\theta_2 \right|^2$$

$$\longrightarrow P_{--} = \frac{1}{2} \cos^2(\theta_2 - \theta_1)$$

PW Problem 6.

The inequality $n(a,\bar{b}) + n(b,\bar{c}) \geq n(a,\bar{c})$ translates to probabilities of photons passing|not passing through the polarizers:

$$P_{+-}(\theta_a - \theta_b) + P_{+-}(\theta_b - \theta_c) \geq P_{+-}(\theta_a - \theta_c)$$

(a) For $\theta_a = 0°$, $\theta_b = 22.5°$, and $\theta_c = 45°$ the inequality evaluates as

$$\frac{1}{2} \sin^2(0° - 22.5°) + \frac{1}{2} \sin^2(22.5° - 45°) \geq \frac{1}{2} \sin^2(0° - 45°)$$

$$\longrightarrow .0732 + .0732 \geq .25 \longrightarrow .146 \geq .25,$$

which is a violation of the inequality.

(b) For $\theta_a = 0°$, $\theta_b = 45°$, and $\theta_c = 90°$ the inequality evaluates as

$$\frac{1}{2} \sin^2(0° - 45°) + \frac{1}{2} \sin^2(45° - 90°) \geq \frac{1}{2} \sin^2(0° - 90°)$$

$$\longrightarrow .25 + .25 \geq .5 \longrightarrow .5 \geq .5,$$

which is satisfies the inequality.