Question 1a. If the signal photon leaves the crystal at 5°, its angle in the crystal θ_c would be:

$$n_o \sin\theta_c = \sin 5°.$$

In class I computed $n_o = 1.6603$ for 810 nm. So,

$$\theta_c = \sin^{-1}\left(\frac{\sin 5°}{1.6603}\right) = 3°.$$

Inside the crystal, conservation of momentum and energy gives

$$\tilde{n}_e = n_o \cos 3°$$

$$\rightarrow \tilde{n}_e = 1.658$$

In class I computed $n_e = 1.5671$ and $n_o = 1.6919$ for a wavelength of 405 nm. Now find the value of θ_m needed, by solving

$$1.658 = \left[\frac{\cos^2\theta_m}{(1.6919)^2} + \frac{\sin^2\theta_m}{(1.5671)^2}\right]$$

Numerically, I find

$$\theta_m = 29.95°$$

Question 1b. If the signal photon leaves the crystal at 0°, its angle in the crystal $\theta_c = 0$, so

$$\tilde{n}_e(405) = n_o(810)$$

$$\rightarrow \tilde{n}_e = 1.6603$$

Again, find the value of θ_m needed, by solving

$$1.6603 = \left[\frac{\cos^2\theta_m}{(1.6919)^2} + \frac{\sin^2\theta_m}{(1.5671)^2}\right]$$

Numerically, I find

$$\theta_m = 28.8°$$

Question 2. A unit vector $|u\rangle$ satisfies $\langle u|u\rangle = 1$. Therefore:

$$\hat{P}_u^2 = |u\rangle\langle u|u\rangle\langle u| = |u\rangle\langle u| = \hat{P}_u.$$

The vector $|u\rangle$ is the eigenvector of $\hat{P}_u$ with eigenvalue 1:

$$\hat{P}_u|u\rangle = |u\rangle\langle u|u\rangle = |u\rangle.$$

Any vector orthogonal to $|u\rangle$ will have eigenvalue zero. Since there exists an orthonormal basis containing $|u\rangle$ as one of the basis vectors, this is enough to completely characterize the operator. The projection operator leaves vector components along the projection axis unchanged and sends orthogonal components to zero.

Question 3.

$$\begin{aligned}\hat{I}|\psi\rangle &= \sum_{i=1}^{n} |e_i\rangle\langle e_i| \sum_{j=1}^{n} c_j|e_j\rangle \\ &= \sum_{i,j} c_j|e_i\rangle\langle e_i|e_j\rangle = \sum_{i,j} c_j|e_i\rangle\delta_{ij} \\ &= \sum_{j} c_j|e_j\rangle = |\psi\rangle.\end{aligned}$$

Question 4. For this problem you want to write out the Taylor series and then sum the terms element by element in the matrices:

$$\begin{aligned}\hat{R}(\theta) = e^{i\theta\hat{l}_z/\hbar} &= 1 + i\theta\hat{l}_z/\hbar + \frac{(i\theta)^2}{2!}\hat{l}_z^2/\hbar^2 + \frac{(i\theta)^3}{3!}\hat{l}_z^3/\hbar^3 + \dots \\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\theta\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} - \frac{\theta^2}{2}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}^2 - i\frac{\theta^3}{6}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}^3 + \dots \\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & -\theta\\ +\theta & 0\end{pmatrix} + \begin{pmatrix}-\frac{\theta^2}{2} & 0\\ 0 & -\frac{\theta^2}{2}\end{pmatrix} + \begin{pmatrix}0 & \frac{\theta^3}{6}\\ -\frac{\theta^3}{6} & 0\end{pmatrix} + \dots \\ &= \begin{pmatrix}1 - \frac{\theta^2}{2} + \dots & -\theta + \frac{\theta^3}{6} + \dots\\ \theta - \frac{\theta^3}{6} + \dots & 1 - \frac{\theta^2}{2} + \dots\end{pmatrix} \\ &= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.\end{aligned}$$

Acting $\hat{R}$ on state $\binom{\cos\phi}{\sin\phi}$ polarized at a generic angle ϕ above the x axis, we can see that it rotates the polarization by angle θ:

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix} = \begin{pmatrix}\cos\theta\cos\phi - \sin\theta\sin\phi\\ \sin\theta\cos\phi + \cos\theta\sin\phi\end{pmatrix} = \begin{pmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\end{pmatrix}$$

Question 5. Solved in class.

Question 6.

(a) In this basis the normalized eigenvectors are as follows:

$$S_x: \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}; \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}; \quad S_y: \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}; \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}; \quad S_z: \begin{pmatrix}1\\ 0\end{pmatrix}; \begin{pmatrix}0\\ 1\end{pmatrix}.$$

With the eigenvectors in hand, it is easy to check that for each operator the corresponding eigenvalues are $\pm\hbar/2$.

(b) For instance,

$$[S_y, S_z] = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \tag{1}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} \tag{2}$$

$$= \frac{i\hbar^2}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\hbar S_x. \tag{3}$$

Question 7.

(a) Done in class.

(b) Call the energy eigenvalues

$$E_{\pm} = \pm \frac{geB_z\hbar}{4m},$$

corresponding to spin up / spin down along the z axis. Then since the state $|\psi(0)\rangle$ is already expanded in the basis of energy eigenstates, we can simple add the oscillating phase factors:

$$|\psi(t)\rangle = \frac{\sqrt{3}}{2} e^{-iE_+ t/\hbar} |z+\rangle + \frac{1}{2} e^{-iE_- t/\hbar} |z-\rangle.$$

The probabilities are:

$$\begin{aligned}
P(z+; t) &= |\langle z + |\psi(t)\rangle|^2 = \left| \frac{\sqrt{3}}{2} e^{-iE_+ t/\hbar} \right|^2 = \frac{3}{4}; \\
P(z-; t) &= \frac{1}{4}; \\
P(y+; t) &= |\langle y + |\psi(t)\rangle|^2 \\
&= \left| \left(\frac{1}{\sqrt{2}} \langle z + | - i \frac{1}{\sqrt{2}} \langle z - |\right)\left(\frac{\sqrt{3}}{2} e^{-iE_+ t/\hbar} |z+\rangle + \frac{1}{2} e^{-iE_- t/\hbar} |z-\rangle\right) \right|^2 \\
&= \left| \frac{\sqrt{3}}{2\sqrt{2}} e^{-iE_+ t/\hbar} - i \frac{1}{2\sqrt{2}} e^{-iE_- t/\hbar} \right|^2 \\
&= \frac{1}{2} - i \frac{\sqrt{3}}{8} (e^{iE_+ t/\hbar - iE_- t/\hbar} - e^{-iE_+ t/\hbar + iE_- t/\hbar}) \\
&= \frac{1}{2} + \frac{\sqrt{3}}{4} \sin((E_+ - E_-)t/\hbar); \\
P(y-; t) &= \frac{1}{2} - \frac{\sqrt{3}}{4} \sin((E_+ - E_-)t/\hbar); \\
P(x+; t) &= \frac{1}{2} + \frac{\sqrt{3}}{4} \cos((E_+ - E_-)t/\hbar); \\
P(x-; t) &= \frac{1}{2} - \frac{\sqrt{3}}{4} \cos((E_+ - E_-)t/\hbar)
\end{aligned}$$

Since there are only two possible spin states along each axis, $P(z+) + P(z-) = 1$, $P(y+) + P(y-) = 1$, and $P(x+) + P(x-) = 1$. Looking at the $P(y+; t)$ values, they fall in the range $\sim [.07, .93]$, from when the spin is mostly anti-aligned with the $+y$ axis to mostly aligned. The spin is tipped up toward $z+$ from the example we studied in class, but the basic behavior is the same – the spin precesses around the magnetic field.