Question 2.1.

Assume N photons are incident on the first polarizer. The incoming photons are randomly polarized, which means they are in a 50/50 mixture of horizontal and vertical polarizations. Since the first polarizer only passes horizontal photons, the number of photons after the first polarizer will be $N/2$, and the photons will all be in the state $|\rightarrow\rangle$. After passing through the second polarizer, photons will be in the state:

$$|\psi\rangle = \cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle$$

Since the initial state is $|\rightarrow\rangle$, the overlap with the final state gives an overlap of

$$\langle\psi|\rightarrow\rangle = \cos\theta,$$

So, the probability of a photon passing through the second polarizer is $\cos^2\theta$. The number of photons after the second polarizer is $N/2\cos^2\theta$.

After passing through the third polarizer, the photons will be in a final state $|\uparrow\rangle$. The overlap of final and initial states is

$$\langle\uparrow|\psi\rangle = \sin\theta,$$

and the probability of a photon passing is $\sin^2\theta$. So, the total number of photons passing through the three polarizers will be:

$$N_3 = \frac{N}{2}\cos^2\theta\sin^2\theta$$

Question 2.2

(a) Any state vector multiplied by a scalar number ($\in \mathbb{C}$) still represents the same quantum state; an overall stretching or phase shift does not change the quantum state. So, $|0\rangle$ and $-|0\rangle$ represent the same quantum state.

(b) See discussion in (a). $|1\rangle$ and $\mathbf{i}\,|1\rangle$ represent the same quantum state.

(c) The states $|0\rangle + |1\rangle$ and $-|0\rangle + \mathbf{i}\,|1\rangle$ are not scalar multiples of each other; they are different superposition states in the fundamental basis. Make a measurement in the left hand circular polarization state:

$$\langle\mathbf{i}|\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{2}(\langle 0| - \mathbf{i}\,\langle 1|)(|0\rangle + |1\rangle) = \frac{1-i}{2}$$

$$\langle\mathbf{i}|\left(\frac{1}{\sqrt{2}}(-|0\rangle + \mathbf{i}\,|1\rangle)\right) = \frac{1}{2}(\langle 0| - \mathbf{i}\,\langle 1|)(-|0\rangle + \mathbf{i}\,|1\rangle) = -\frac{1-i^2}{2} = 0$$

Note that the probability for the first measurement is $\frac{1}{2}|1-i|^2 = 1$, while for the second it is zero.

(d) The states $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ are not scalar multiples of each other; they are different superposition states in the fundamental basis. Make a measurement in the Hadamard basis:

$$\langle +|\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle + |1\rangle) = 1$$

$$\langle +| \left(\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right) = \frac{1}{2} (\langle 0| + \langle 1|)(|0\rangle - |1\rangle) = 0$$

(e) The states $|0\rangle - |1\rangle$ and $|1\rangle - |0\rangle$ are scalar multiples of each other:

$$\frac{1}{\sqrt{2}} (|1\rangle - |0\rangle) = -\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

So, the two states differ by an global phase of -1 and represent the same state.

(f) The states are the right and left polarization states modulo a global phase, so they are orthogonal:

$$\frac{1}{\sqrt{2}} (|0\rangle + \mathbf{i} |1\rangle) = |\mathbf{i}\rangle$$

$$\frac{1}{\sqrt{2}} (\mathbf{i} |1\rangle - |0\rangle) = -\frac{1}{\sqrt{2}} (|0\rangle - \mathbf{i} |1\rangle) = - |-\mathbf{i}\rangle$$

(g) These two states are identical:

$$\frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right) = \frac{1}{2} (2 |0\rangle) = |0\rangle$$

(h) These two states are identical, up to a global phase:

$$\frac{1}{\sqrt{2}} (|\mathbf{i}\rangle - |-\mathbf{i}\rangle) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (|0\rangle + \mathbf{i} |1\rangle) - \frac{1}{\sqrt{2}} (|0\rangle - \mathbf{i} |1\rangle) \right) = \frac{\mathbf{i}}{2} (2 |1\rangle) = \mathbf{i} |1\rangle$$

(i) These two states are identical:

$$\frac{1}{\sqrt{2}} (|\mathbf{i}\rangle + |-\mathbf{i}\rangle) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (|0\rangle + \mathbf{i} |1\rangle) + \frac{1}{\sqrt{2}} (|0\rangle - \mathbf{i} |1\rangle) \right) = \frac{1}{2} (2 |0\rangle) = |0\rangle$$

$$\frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right)$$

(j) These two states are the same, up to a global phase:

$$\frac{1}{\sqrt{2}} \left(e^{-\frac{i\pi}{4}} |0\rangle + |1\rangle \right) = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}} \left(|0\rangle + e^{\frac{i\pi}{4}} |1\rangle \right)$$

Question 2.3

(a) This state vector is a pure state (no superposition) in the Hadamard basis ($\{|+\rangle, |-\rangle\}$). It is a superposition in the standard basis:

$$|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

(b) Expand in the standard basis:

$$\frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right) = |0\rangle .$$

So, it is a pure state in the standard basis, $\{|0\rangle, |1\rangle\}$.

(c) Expand in the standard basis:

$$\frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) - \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right) = |1\rangle .$$

So, it is a pure state in the standard basis, $\{|0\rangle, |1\rangle\}$.

(d) Expand in the standard basis:

$$\begin{aligned}\frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle &= \tfrac{\sqrt{3}}{2}\left(\tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right) + \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right) \\ &= \tfrac{\sqrt{3}+1}{2\sqrt{2}}|0\rangle + \tfrac{\sqrt{3}+1}{2\sqrt{2}}|1\rangle \\ &= \cos 15^\circ |0\rangle + \sin 15^\circ |1\rangle\end{aligned}$$

So, this is a superposition with respect to the standard basis. If we choose a basis constructed with an axis at 15° from the horizontal axis:

$$\{|u\rangle, |v\rangle\} = \{\cos 15^\circ |0\rangle + \sin 15^\circ |1\rangle, -\sin 15^\circ |0\rangle + \cos 15^\circ |1\rangle\},$$

Then the state given is the pure state $|u\rangle$. Note that $\{|u\rangle, |v\rangle\}$ is an orthonormal basis.

(e) See part (h) in 2.2:

$$\frac{1}{\sqrt{2}}\left(|\mathbf{i}\rangle - |-\mathbf{i}\rangle\right) = \mathbf{i}\,|1\rangle$$

So this is a pure state in the standard basis.

(f) By inspection this is a superposition in the standard basis. And

$$|-\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right),$$

so this state is not a superposition in the Hadamard basis.

Question 2.4

By inspection, I immediately see that (b), (c) and (d) are superpositions in the Hadamard basis; and (a) is not. I also showed that (f) was not a superposition in the Hadamard basis. Finally, since (e) is a pure state in the standard basis, it must be a superposition state in the Hadamard basis.

Question 2.5

(a) Expand the Hadamard states into the standard basis (See (c) in 2.3):

$$\frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) = |1\rangle,$$

so

$$e^{i\theta} = -1,$$

which is true for $\theta = \{\pm\pi, \pm 3\pi, \pm 5\pi ...\}$

(b) These two states only differ by a global phase:

$$\frac{1}{\sqrt{2}}\left(|-\mathbf{i}\rangle + e^{-i\theta}|\mathbf{i}\rangle\right) = \frac{e^{-i\theta}}{\sqrt{2}}\left(e^{i\theta}|-\mathbf{i}\rangle + |\mathbf{i}\rangle\right) = \frac{e^{-i\theta}}{\sqrt{2}}\left(|\mathbf{i}\rangle + e^{i\theta}|-\mathbf{i}\rangle\right)$$

so all values of θ work.

(c) By inspection, the two states only differ by a global phase, so any value of θ will work.

Question 2.6
If we have the state expressed in the measurement basis, then the probability of a measurement outcome is given by the coefficient of each state squared.
(a) The probability of measuring the photon in the state $|0\rangle$ is

$$P_0 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4},$$

and the probability of finding the photon in the state $|1\rangle$ is

$$P_1 = \left|-\frac{1}{2}\right|^2 = \frac{1}{4},$$

Note that

$$P_0 + P_1 = 1$$

(b) The probability of measuring the photon in the state $|0\rangle$ is

$$P_0 = \left|-\frac{1}{2}\right|^2 = \frac{1}{4},$$

and the probability of finding the photon in the state $|1\rangle$ is

$$P_1 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4},$$

(c) To find the probability amplitudes, express the state in terms of the standard basis vectors:

$$|-\mathbf{i}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - \mathbf{i}\,|1\rangle\right)$$

using the expansion coefficients:

$$P_0 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

and the probability of finding the photon in the state $|1\rangle$ is

$$P_1 = \left|\frac{-\mathbf{i}}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

(d) To find the probability amplitudes, express the state in terms of the Hadamard basis:

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

using the expansion coefficients:

$$P_+ = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

and the probability of finding the photon in the state $|1\rangle$ is

$$P_{-} = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

(e) To find the probability amplitudes, express the state in terms of the circular polarization basis:

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{i}\rangle + |-\mathbf{i}\rangle\right)$$

$$|1\rangle = -\frac{\mathbf{i}}{\sqrt{2}}\left(|\mathbf{i}\rangle - |-\mathbf{i}\rangle\right)$$

So,

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) &= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\left(|\mathbf{i}\rangle + |-\mathbf{i}\rangle\right) - \tfrac{\mathbf{i}}{\sqrt{2}}\left(|\mathbf{i}\rangle - |-\mathbf{i}\rangle\right)\right) \\ &= \tfrac{1-\mathbf{i}}{2}|\mathbf{i}\rangle + \tfrac{1+\mathbf{i}}{2}|-\mathbf{i}\rangle\end{aligned}$$

Measurements yield either a photon in the $|\mathbf{i}\rangle$ state or the $|-\mathbf{i}\rangle$ state:

$$P_{\mathbf{i}} = \left|\frac{1-\mathbf{i}}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$$

$$P_{-\mathbf{i}} = \left|\frac{1+\mathbf{i}}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$$

(f) To find the probability amplitudes, express the state in terms of the circular polarization basis (see previous problem):

$$|1\rangle = -\frac{\mathbf{i}}{\sqrt{2}}\left(|\mathbf{i}\rangle - |-\mathbf{i}\rangle\right)$$

Measurements yield either a photon in the $|\mathbf{i}\rangle$ state or the $|-\mathbf{i}\rangle$ state:

$$P_{\mathbf{i}} = \left|\frac{-\mathbf{i}}{2}\right|^2 = \frac{1}{2}$$

$$P_{-\mathbf{i}} = \left|\frac{\mathbf{i}}{2}\right|^2 = \frac{1}{2}$$

(g) I need to express $|+\rangle$ in terms of the measurement states given:

$$|s\rangle \equiv \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$|t\rangle \equiv \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$$

Then write the Hadamard state $|+\rangle$ in terms of these basis states. I find

$$|0\rangle = \frac{1}{2}|s\rangle + \frac{\sqrt{3}}{2}|t\rangle$$

$$|1\rangle = \frac{\sqrt{3}}{2}|s\rangle - \frac{1}{2}|t\rangle$$

I can use these to write the Hadamard state in terms of the measurement basis:

$$\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{2}|s\rangle + \frac{\sqrt{3}}{2}|t\rangle + \frac{\sqrt{3}}{2}|s\rangle - \frac{1}{2}|t\rangle\right) \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}}|s\rangle + \frac{\sqrt{3}-1}{2\sqrt{2}}|t\rangle
\end{aligned}$$

Using the coefficients, the probability of measurement for the two possible outcomes $|s\rangle$ and $|t\rangle$ are

$$P_s = \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{4+2\sqrt{3}}{8}$$

$$P_t = \left|\frac{\sqrt{3}-1}{2\sqrt{2}}\right|^2 = \frac{4-2\sqrt{3}}{8}$$

Note the probabilities add to 1:

$$P_s + P_t = \frac{4+2\sqrt{3}}{8} + \frac{4-2\sqrt{3}}{8} = \frac{4+4}{8} = 1$$

Question 2.7

To find the orthonormal basis that a particular state vector belongs to, we must identify a normalized vector that is orthogonal ("perpendicular") to the given vector. We can accomplish this with the inner product. Any normalized vector can be multiplied by a global phase $e^{-\mathbf{i}\alpha}$ ($\alpha \in \mathbb{R}$) and it is still normalized and gives the same state. I will suppress this phase in the answers below, but it can be added. This means there are an infinitely many choices.

(a)

$$|\mathbf{i}\rangle = \frac{1}{2}(|0\rangle + \mathbf{i}|1\rangle),$$

So the vector orthogonal to this is $|-\mathbf{i}\rangle$. This means the orthonormal basis for this state is

$$\left\{|\mathbf{i}\rangle = \frac{1}{2}(|0\rangle + \mathbf{i}|1\rangle), |-\mathbf{i}\rangle = \frac{1}{2}(|0\rangle - \mathbf{i}|1\rangle)\right\}$$

(b) The factors used in representing this vector can be written as

$$\frac{1+\mathbf{i}}{2} = \frac{e^{\mathbf{i}\frac{\pi}{4}}}{\sqrt{2}}$$

$$\frac{1-\mathbf{i}}{2} = \frac{e^{-\mathbf{i}\frac{\pi}{4}}}{\sqrt{2}}$$

So,

$$\frac{1+\mathbf{i}}{2}|0\rangle - \frac{1-\mathbf{i}}{2}|1\rangle = \frac{1}{\sqrt{2}}\left(e^{\mathbf{i}\frac{\pi}{4}}|0\rangle - e^{-\mathbf{i}\frac{\pi}{4}}|1\rangle\right) = \frac{e^{\mathbf{i}\frac{\pi}{4}}}{\sqrt{2}}\left(|0\rangle - e^{-\mathbf{i}\frac{\pi}{2}}|1\rangle\right) = \frac{e^{\mathbf{i}\frac{\pi}{4}}}{\sqrt{2}}\left(|0\rangle + \mathbf{i}|1\rangle\right)$$

This means the same orthonormal bases as in (a) work.

(c) I want to find a vector $a|0\rangle + b|1\rangle$ that is orthogonal to the given state:

$$\left(a^*\langle 0| + b^*\langle 1|\right)\left(\frac{1}{\sqrt{2}}\left(|0\rangle + e^{\mathbf{i}\frac{\pi}{6}}|1\rangle\right)\right) = 0$$

This means

$$a^* + b^* e^{\mathbf{i}\frac{\pi}{6}} = 0 \longrightarrow a = -b e^{-\mathbf{i}\frac{\pi}{6}}$$

The normalization condition yeilds:

$$\left|-b e^{-\mathbf{i}\frac{\pi}{6}}\right|^2 + |b|^2 = 1 \longrightarrow |b| = \frac{1}{\sqrt{2}}$$

Note, we can only determine the value of b up to a pure phase (which would be a global phase for the state we are looking for). So, I can write the orthonormal vector as

$$\frac{1}{\sqrt{2}}\left(-e^{-\mathbf{i}\frac{\pi}{6}}|0\rangle + |1\rangle\right)$$

You can check that this is normalized and orthogonal to $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{\mathbf{i}\frac{\pi}{6}}|1\rangle\right)$. So, the orthogonal basis (again, modulo a global phase) is

$$\left\{\frac{1}{\sqrt{2}}\left(-e^{-\mathbf{i}\frac{\pi}{6}}|0\rangle + |1\rangle\right), \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\mathbf{i}\frac{\pi}{6}}|1\rangle\right)\right\}$$

(d) Repeat the process in (c):

$$\left(a^*\langle +| + b^*\langle -|\right)\left(\frac{1}{2}|+\rangle - \frac{\mathbf{i}\sqrt{3}}{2}|-\rangle\right) = 0$$

This means

$$\frac{a^*}{2} - \frac{ib^*\sqrt{3}}{2} = 0 \longrightarrow a = -\mathbf{i}b\sqrt{3}$$

The normalization condition yeilds:

$$\left|-\mathbf{i}b\sqrt{3}\right|^2 + |b|^2 = 1 \longrightarrow |b| = \frac{1}{2}$$

Putting the solutions for a and b in, I get the orthonormal vector

$$-\mathbf{i}\frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle$$

So, the orthogonal basis (again, modulo a global phase) is

$$\left\{\frac{1}{2}|+\rangle - \frac{\mathbf{i}\sqrt{3}}{2}|-\rangle, -\mathbf{i}\frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle\right\}$$

Question 2.8 The difference between $|1\rangle$ and $-|1\rangle$ is a global phase. Multiplying a state by an overall complex constant does not change the "direction" or the quantum state. So, if we took the state $|0\rangle - |1\rangle$ and multiplied it by -1:

$$\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \longrightarrow -\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right),$$

the state would remain unchanged; it would still describe the same quantum state. The sum of two states is a quantum mechanical superposition of states and the global versus relative phase makes a difference in the state. Thus, $\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$ is not the same quantum state as $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$. These two states differ by a relative phase–they cannot be written a multiple of one another.