Physics 140 QoR - Midterm solutions

I. Short answer

The responses below include suggestions (not exhaustive) of points you might have raised in answering the questions.

1. A hallmark of a wave is interference. The amplitudes of two waves add, and the intensity (classically) or probability (quantum mechanically) is given by the square of the amplitudes:

   $$|\phi_{tot}|^2 = |\phi_1 + \phi_2|^2 = |\phi_1|^2 + |\phi_2|^2 + 2\mathrm{Re}(\phi_1 \phi_2^*).$$

   You can read ϕ in this equation as a generic wave amplitude, for instance, the wavefunction in quantum mechanics or a component of the classical electric field. The cross terms $2\mathrm{Re}(\phi_1\phi_2^*)$ can be positive or negative, either amplifying or cancelling the summed intensities of the two individual waves, depending on whether at a given point the two waves are in or out of phase. A surprising feature of interference is that by introducing more sources, you can get less intensity at certain places than you would find with one source alone. We studied wave behavior most notably in the electron two-slit interference experiment, which is the subject of problem two....

   One hallmark of a particle is that it comes in lumps: It will be detected at a well-localized point in space and in time. We noted this behavior in the build up of the electron two-slit interference pattern, in which detections registered as discrete points of light on the scintillation screen. If you have two sources of particles, the intensities (rate of particle arrival) add.

   Alternately, you could have discussed the sense in which a particle is indivisible, e.g. our experiment with single photons in a beamsplitter.

2. You see an interference pattern. I was looking for a picture like Figure 3 in the Bach paper, which captures both the interference pattern and the discrete nature of individual photon detections – not a sketch of an intensity curve but the arrangement of accumulated flashes you would actually observe on a scintillation screen.

   If you were to move the screen up so that the distance from slits to screen was much less than the separation between the slits, negligible wave amplitude from slit A would reach the detector behind slit B, and vice-versa, so there would be no interference. You would see two distinct aggregations of flashes of light. You might also have mentioned that the screen resolves the position of the electron – which slit it traversed – thereby collapsing the wavefunction and killing the interference pattern.

3. To treat the photoelectric effect semi-classically is to develop a hybrid theory in which the matter is treated within quantum mechanics and the

light is modeled as a classical Maxwellian electromagnetic wave. In time-dependent perturbation theory we developed the following result for the probability an electron is ejected from a solid by an incoming electromagnetic wave, of amplitude $\mathcal{E}_0$ and frequency ω, into the continuum with density of states $g(E)$:

$$P(t) = \frac{\pi}{2\hbar}\left(e\mathcal{E}_0\right)^2 \left|\chi_{fg}(\hbar\omega - E_I)\right|^2 g(\hbar\omega - E_I)t.$$

Here χ_{fg} is the matrix element of the $\hat{x}$ operator between initial and final states and E_I is the ionization energy. The frequency dependence of the stopping potential can be seen in this expression through the density of states factor $g(\hbar\omega - E_I)$, which is only non-zero for $\hbar\omega > E_I$. It is the quantity $\hbar\omega$ which must exceed the ionization energy for electrons to be liberated. Or you might trace this behavior back to the sinc function at an earlier stage of the derivation. By contrast, the intensity of light (proportional to $\mathcal{E}_0^2$) appears only as a constant factor out front. This question is begging for an equation.

4. The thermal light used in the Hanbury-Brown and Twiss experiment is bunched light, obeying Bose statistics: The state of the field is not a photon number eigenstate, and photons tend to arrive in clumps. HB&T found coincidenes not because any individual photon split in the beamsplitter, but because they detected mulitple photons in their apparatus at once.

II. Computation

1-4. See pages below.

5. For the hydrogen atom the Hamiltonian for the nuclear spin in the external field $\vec{B} = B_0\hat{k}$ is

$$\hat{H} = -\hat{\vec{\mu}}_H \cdot \vec{B} = -\frac{g_H e}{2m_p}\hat{\vec{I}} \cdot \vec{B} = -\frac{g_H e B}{2m_p}\hat{I}_z.$$

   Since this is a spin-1/2 nucleus the eigenvalues of I_z are $\pm\hbar/2$, giving energies

$$E_{\mp} = \pm\frac{g_H e B\hbar}{4m_p}.$$

   As you observed in the previous problem, resonant absorption occurs when the oscillating field has frequency equal to the energy difference between the two states (divided by $\hbar$):

$$\omega = \Delta E/\hbar = (E_- - E_+)/\hbar = \frac{g_H e B}{2m_p}.$$

   Plugging in,

$$\omega = \frac{g_H e B}{2m_p} = \frac{5.58(1.60 \cdot 10^{-19}\,\mathrm{C})(3\,\mathrm{T})}{2 \cdot 1.67 \cdot 10^{-27}\,\mathrm{kg}} = 8.02 \cdot 10^8\,\mathrm{rad/s};$$
$$\nu = \frac{\omega}{2\pi} = 128\,\mathrm{MHz}.$$

① Computation 1

$$|x+\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle + |z-\rangle\right)$$

$$|x-\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle - |z-\rangle\right)$$

≈ normalized

$$P(x+) = |\langle x+|\psi\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}\left(\langle z+| + \langle z-|\right)\left(.89|z+\rangle + .45|z-\rangle\right)\right|^2$$

$$= \left|\frac{.89}{\sqrt{2}} + \frac{.45}{\sqrt{2}}\right|^2 = \boxed{.90}$$

Note $$P(x-) = |\langle x-|\psi\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}\left(\langle z+| - \langle z-|\right)\left(.89|z+\rangle + .45|z-\rangle\right)\right|^2$$

$$= \left|\frac{.89}{\sqrt{2}} - \frac{.45}{\sqrt{2}}\right|^2 \simeq \boxed{.1}$$

$$\left.\begin{aligned} |y+\rangle &= \frac{1}{\sqrt{2}}\left(|z+\rangle + i\,|z-\rangle\right) \\ |y-\rangle &= \frac{1}{\sqrt{2}}\left(|z+\rangle - i\,|z-\rangle\right) \end{aligned}\right\} \rightarrow \begin{aligned} |z+\rangle &= \frac{1}{\sqrt{2}}\left(|y+\rangle + |y-\rangle\right) \\ |z-\rangle &= \frac{-i}{\sqrt{2}}\left(|y+\rangle - |y-\rangle\right) \end{aligned}$$

$$P(y+) = |\langle y+|\psi\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}\left(\langle z+| - i\langle z-|\right)\left(.89|z+\rangle + .45|z-\rangle\right)\right|^2$$

$$= \left|\frac{.89}{\sqrt{2}} - \frac{i(.45)}{\sqrt{2}}\right|^2 = \left(\frac{.89}{\sqrt{2}}\right)^2 + \left(\frac{.45}{\sqrt{2}}\right)^2 = \boxed{.50}$$

Initial State:

$$|\theta\rangle = \cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle$$

Final State:

$$|\phi\rangle = \cos\phi\,|\rightarrow\rangle + \sin\phi\,|\uparrow\rangle$$

$$P(\theta \rightarrow \phi) = \left|\langle\phi|\theta\rangle\right|^2$$

$$= \left|\left(\cos\phi\,\langle\rightarrow| + \sin\phi\,\langle\uparrow|\right)\left(\cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle\right)\right|^2$$

$$= \left|\cos\phi\cos\theta + \sin\phi\sin\theta\right|^2$$

$$= \left|\cos(\theta-\phi)\right|^2$$

$$= \boxed{\cos^2(\theta-\phi)}$$

Computation 3

Final state:

$$|\phi\rangle = \cos\phi\,|\rightarrow\rangle + \sin\phi\,|\uparrow\rangle$$

Initial state:

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle - i\,|\uparrow\rangle\right)$$

$$P(\phi) = \left|\langle\phi|R\rangle\right|^2$$

$$= \frac{1}{2}\left|\left(\cos\phi\,\langle\rightarrow| + \sin\phi\,\langle\uparrow|\right)\left(|\rightarrow\rangle - i\,|\uparrow\rangle\right)\right|^2$$

$$= \frac{1}{2}\left|\cos\phi - i\sin\phi\right|^2$$

$$= \frac{1}{2}\left|e^{-i\phi}\right|^2$$

$$= \boxed{\frac{1}{2}}$$

Time dependent P-T says:

$$\frac{dc_m^{(1)}}{dt} = \frac{-i}{\hbar} \sum c_n^{(0)}(t) \langle \phi_m | H' | \phi_n \rangle e^{\frac{-i}{\hbar}(E_n - E_m)t}$$

The probability of transition from initial state i to final state m is:

$P_{i \to m} = |c_m|^2$ with $c_n^{(0)}$'s determined by the initial state

In our problem:

$$H' = -\vec{\mu} \cdot (B_x \cos\omega t \, \hat{i})$$

$$= \frac{+ge}{2m} S_x B_x \cos\omega t$$ ✱

Initial state: $c_n^{(0)}(t) = c_-(t)$

$= 1$ (since eigenstate)

First ~~compute~~ matrix element:

final state is $|z+\rangle$

$$\therefore \quad \langle \phi_m | H' | \phi_n \rangle \equiv \langle z+ | \frac{ge}{2m} B_x \cos\omega t \, S_x | z- \rangle$$

$$= \frac{geB_x \cos\omega t}{2m} \langle z+ | S_x | z- \rangle$$ ✱

$$\langle z+|S_x|z-\rangle = (1\ 0)\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \left(0\ \ \tfrac{\hbar}{2}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \hbar/2 \qquad \text{✯}$$

$$\therefore \frac{dc_+^{(1)}}{dt} = \frac{-i}{\hbar}\left(\frac{geB_x\cos\omega t}{2m}\right)\left(\frac{\hbar}{2}\right) e^{\frac{-i}{\hbar}(E_- - E_+)t}$$

define $\dfrac{E_- - E_+}{\hbar} \equiv \Delta\omega_\pm$

$$\therefore \frac{dc_+^{(1)}}{dt} = \frac{-i}{2}\left(\frac{geB_x}{2m}\right) e^{-i\Delta\omega_\pm t}\cos\omega t \qquad \text{✯}$$

Integrate $t' = 0$ to t $\quad [H' = 0 \text{ for } t < 0]$

$$\therefore C_+^{(1)} = \frac{-igeB_x}{4m}\underbrace{\int_0^t e^{-i\Delta\omega_\pm t'}\cos\omega t'\,dt'}_{\equiv I} \qquad \text{✯}$$

$$I = \frac{1}{2}\int_0^t e^{-i\Delta\omega_\pm t}\left[e^{i\omega t} + e^{-i\omega t}\right]dt$$

$$= \frac{1}{2}\left[\frac{e^{-i(\Delta\omega_\pm - \omega)t}}{-i(\Delta\omega_\pm - \omega)} + \frac{e^{-i(\Delta\omega_\pm + \omega)t}}{-i(\Delta\omega_\pm + \omega)}\right]\Bigg|_0^t$$

$$= \frac{1}{2}\left[\frac{e^{-i(\Delta\omega_\pm - \omega)t} - 1}{-i(\Delta\omega_\pm - \omega)} + \frac{e^{-i(\Delta\omega_\pm + \omega)t} - 1}{-i(\Delta\omega_\pm + \omega}\right]$$

$\star$

recall : highly peaked $\Delta\omega_\pm \approx \omega$

$$\rightarrow \quad I = \frac{1}{2}\left(\frac{e^{-i(\Delta\omega_\pm - \omega)t} - 1}{-i(\Delta\omega_\pm - \omega)}\right)$$

$$\Rightarrow \quad c_+^{(1)} = \frac{-ig e B_x}{4m} I(t)$$

$$P_{-\to+} = \left|c_+^{(1)}\right|^2 = \left(\frac{geB_x}{4m}\right)^2 I^2(t)$$

$$I^2(t) = \frac{1}{4}\frac{1}{(\Delta\omega_\pm - \omega)^2}\cdot\left|e^{-i(\Delta\omega_\pm - \omega)t} - 1\right|^2$$

$$= \frac{\sin^2(\Delta\omega_\pm - \omega)t/2}{(\Delta\omega_\pm - \omega)^2}$$

$\star$

$$\therefore \boxed{P_{-\to+} = \left(\frac{geB_x}{4m}\right)^2 \frac{\sin^2(\Delta\omega_{\pm}-\omega)t/2}{(\Delta\omega_{\pm}-\omega)^2}} \quad ☆$$

at ~~near~~ resonance use

$$\left(\frac{t}{2}\right)^2 \frac{\sin^2(\Delta\omega_{\pm}-\omega)t/2}{(\Delta\omega_{\pm}-\omega)(t/2)^2} = \frac{t^2}{4}$$

$$\therefore \boxed{P_{-\to+} = \left(\frac{geB_x}{8m}\right)^2 t^2} \quad \Delta\omega_{\pm} = \omega \quad ☆$$