Thermal properties of the lattice

1/ See your notes. Keep in mind that "stationary" means $v = \Delta v = 0$. The fact that atoms are in the lattice means $\Delta x \approx 1\,nm$ (that is, it's somewhere between 2 atoms).

2/ (a,b) $M \frac{dx^2}{dt^2} = -\gamma x \qquad x(t) = A\cos(\omega t - \delta)$

$$-A\omega^2 M\cos(\omega t - \delta) = -\gamma A\cos(\omega t - \delta)$$

this equality holds if

$$M\omega^2 = \gamma$$

therefore $\omega = \sqrt{\frac{\gamma}{M}}$ the frequency depends on the mass + the force constant.

(c) $E = \frac{1}{2}\gamma A^2 = k_B T$

$$A = \sqrt{\frac{2k_B T}{\gamma}}$$

(d) $\omega = \sqrt{\frac{\gamma}{M}} \approx \sqrt{\frac{10}{10^{-27}}} \approx \sqrt{10^{+28}} \approx 10^{14}\ \frac{rad}{s}$

$\approx 10^{13}\ Hz$ } $\omega = 2\pi f$

$$A = \sqrt{\frac{2(10^{-23})(300)}{10}} \approx \sqrt{6 \cdot 10^{-22}} \approx 10^{-11}\ m$$

3/ (a) Travels to the <u>left</u> $(-x)$ dir.

$f(kx - wt)$ travels to the right

$f(kx + wt)$ left

(b) $3.5 \sin\left(2.7x + 4t - \frac{\pi}{8}\right)$

$2.7 \to k$, $4 \to w$, $-\frac{\pi}{8} \to \phi$

$k = 2.7/m$ wave number

$k = \frac{2\pi}{\lambda} \rightarrow \lambda = \frac{2\pi}{k} = \frac{2.\pi}{2.7} = 2.3m$ wave length

$w = 4$ rad/s

$w = 2\pi f \rightarrow f = \frac{w}{2\pi} = \frac{4}{2\pi} = 0.64$ Hz

$V = \frac{w}{k} = \frac{4}{2.7} = 1.5$ m/s

4/ Consider the series expansion for e^x, $\sin x$ and $\cos x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}$$

$$\sin x = + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

Now consider the case that $x = i\theta$, a complex number when $i = \sqrt{-1}$ such that $i^2 = -1$

then $e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \dots$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \pm \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

5/ $m\frac{d^2u_n}{dt^2} = -\gamma\,(2u_n - u_{n-1} - u_{n+1})$

if $u_n = u e^{i(kan - \omega t)}$

then $u_{n-1} = u e^{i[ka(n-1) - \omega t]}$

$u_{n+1} = u e^{i[ka(n+1) - \omega t]}$

$$\frac{d^2u_n}{dt^2} = -\omega^2 u e^{i(kan - \omega t)}$$

sub these in to find

• $m(-\omega^2) u e^{ik(xan - \omega t)} = -\gamma\left[2ue^{i\gamma(kan - \omega t)} - ue^{i(ka(n-1) - \omega t]} - ue^{i(ka(n+1) - \omega t)}\right]$

all terms have $e^{i(kxan - \omega t)}$. factor this out & cancel

• $-m\omega^2 = -\gamma\left(2 - e^{-ika} - e^{+ika}\right)$ now $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$= -\gamma\left(2 - 2\cos(ka)\right)$$

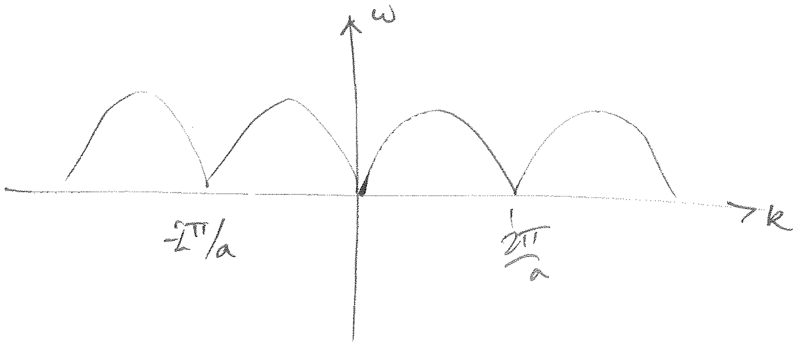
$$= -2\gamma\left(1 - \cos(ka)\right)$$

use the half angle form $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$$\omega^2 = \frac{4\gamma}{m}\sin^2\left(\frac{ka}{2}\right)$$

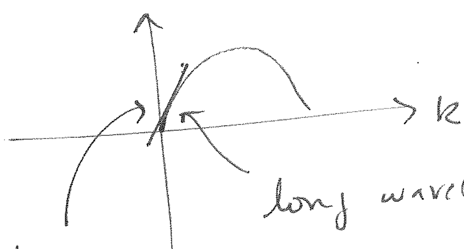
6/

(a)



(b) the period is $\frac{2\pi}{a}$. Note that $\left|\sin\frac{ka}{2}\right|$ doesn't require 2π to complete a cycle → only π is needed i.e. $\sin\left(\frac{2\pi}{a}\frac{a}{2}\right) = \sin(\pi)$.

(c)



$k = \frac{2\pi}{\lambda}$

$k \to 0$

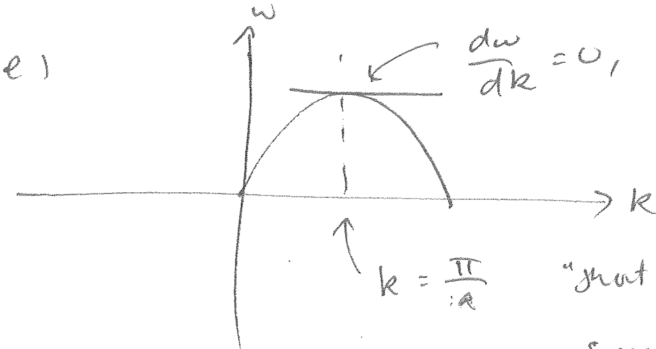
(d) $$\frac{d\omega}{dk} = 2\sqrt{\frac{\gamma}{m}}\left[\cos\left(\frac{ka}{2}\right)\right]\frac{a}{2}$$

for $k \to 0$, $\cos\left(\frac{ka}{2}\right) \to 1$

$$\frac{d\omega}{dk} \approx a\sqrt{\frac{\gamma}{m}}$$

6 cont'd /

(e)



"short wavelength"

since $k = \frac{\pi}{\lambda}$, this is $\lambda = 2a$

any shorter & we wouldn't be able to see the cycle. (or it's just represented by a longer wavelength)

(f)

$$\frac{d\omega}{dk} = 2\sqrt{\frac{\gamma}{m}}\left(\frac{a}{2}\right)\cos\left(\frac{ka}{2}\right)$$

$$k = \frac{\pi}{a}$$

$$\frac{d\omega}{dk} = a\sqrt{\frac{\gamma}{m}}\,\underbrace{\cos\left(\frac{\pi}{a}\cdot\frac{a}{2}\right)}_{\cos\left(\frac{\pi}{2}\right)} = 0$$

7/ (a) $\cos(ka)$ has a full cycle when $k=0$, $\theta=0$ to $k=\frac{2\pi}{a}$, $\theta=2\pi$

$\cos(2ka)$ completes a cycle when $k=0$, $\theta=0$; $k=\frac{\pi}{a}$, $\theta=2\pi$

since we're considering $M\omega^2 = \ldots \cos(ka) + \ldots \cos(\frac{ka}{2})$

the longest period determines the period of $M\omega^2$

$\Rightarrow k = \frac{2\pi}{a}$

(b) $k \to 0$ limit

$\cos(x) \to 1 - \frac{x^2}{2!} \ldots$

$$M\omega^2 = 2\gamma_1\left[1 - (1 - (ka)^2\tfrac{1}{2})\right] + 2\gamma_2\left[1 - (1 - (2ka)^2\tfrac{1}{2}\right]$$

$$= 2\left[\gamma_1\left(+\frac{k^2a^2}{2}\right) + \gamma_2\left(+\frac{4k^2a^2}{2}\right)\right]$$

$$M\omega^2 = \gamma_1(ka)^2 + \gamma_2 4(ka)^2$$

$$\boxed{\omega = k\left(\frac{\gamma_1}{M} + \frac{4\gamma_2}{M}\right)^{1/2}}$$

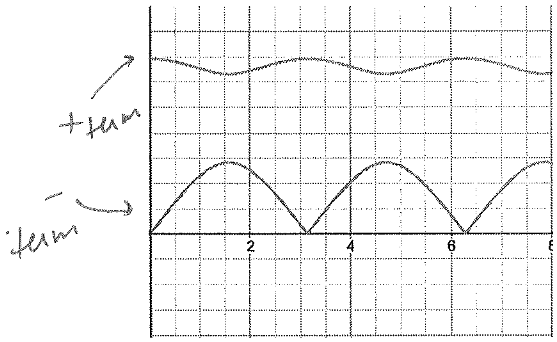
of the form $\omega = vk$ where $v = \left(\frac{\gamma_1}{M} + \frac{4\gamma_2}{M}\right)^{1/2}$

(c) $$\omega = \left\{\frac{2\gamma_1}{M}(1-\cos(ka)] + 2\gamma_2(1-\cos(2ka))]\right\}^{1/2}$$

take $\frac{d\omega}{dk}$ and then set $k = \frac{\pi}{a}$

(d) $k = \frac{\pi}{a} = \frac{2\pi}{\lambda}$ here, $\lambda = 2a$

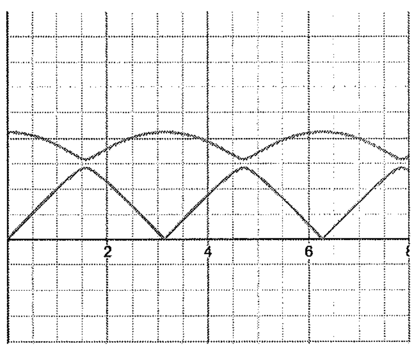
8/ $M_2 = 0.2 M_1$



The more different the two masses are, the more distinct the optical + acoustic branches become. When the masses become the same, the optical + acoustic branches merge together.

(b) $\omega^2 = \gamma\left(\frac{1}{M} + \frac{1}{M}\right) \pm \gamma\left[\left(\frac{1}{M}+\frac{1}{M}\right)^2 - \frac{4}{M^2}\sin^2\frac{kb}{2}\right]^{1/2}$

$M_2 = 0.8 M_1$



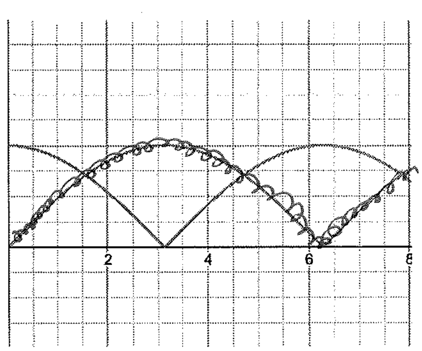
$= \frac{2\gamma}{M} \pm \gamma\left[\left(\frac{2}{M}\right)^2 - \frac{4}{M^2}\sin^2\frac{kb}{2}\right]^{1/2}$

$= \ldots \pm \frac{2\gamma}{M}\left[1 - \sin^2\left(\frac{kb}{2}\right)\right]^{1/2}$

$= \ldots \pm \ldots \left[\cos^2\left(\frac{kb}{2}\right)\right]^{1/2}$

$= \ldots \pm \ldots \cos\left(\frac{kb}{2}\right)$

$M_2 = M_1$



$\omega^2 = \frac{2\gamma}{M}\left[1 \pm \cos\left(\frac{kb}{2}\right)\right]$ $\frac{b}{2} = a$

$\omega = \left[\frac{2\gamma}{M}\left(1 \pm \cos(ka)\right)\right]^{1/2}$

9/ (a) optical refers to its possible coupling w/ EM radiation

acoustic refers to the sound wave characteristics
(long wavelength, linear dispersion)

(b) acoustic... atoms are in phase

if it were a transverse wave,
the atoms would be part of a single wavefunc

optical neighboring atoms are out of phase

(c) infrared

10/ (a) a quantum of light

(b) $E = hf$

(c) $p = h/\lambda$

(d) A beam of light of 3 mW + wavelength 500 nm
Consists of photons w/ energy $E = hf = \frac{hc}{\lambda} = 3.99 \cdot 10^{-19} J$
$= 2.5 eV$

To have a power of 3 mW,

$$\frac{dn}{dt} E_{photon} = \text{power}$$

$$\frac{dn}{dt} = \frac{3 \cdot 10^{-3} J/s}{3.99 \cdot 10^{-19} J/photon} = 7.5 \cdot 10^{15} \frac{photons}{sec}$$

there are $10^{15} \frac{photons}{sec}$ created by the light source

11/ (a) quantum of energy associated with vibrations (b) $E = \hbar\omega$ (c) $p = \hbar/k$