P1.1

We want to use Gauss' law

$$\int E \cdot dA = \frac{Q_{enc}}{\epsilon}$$

We choose a surface that is a cylinder of radius r and length L.

Because of the symmetry of the charge distribution we know that the field is radial. Thus there is no flux through the ends of the surface. Further the field is normal to the curved surface and uniform over the surface.

So $\oint E \cdot dA = \int E\, dA = EA = E\, 2\pi r L$

and $E = \frac{1}{2\pi r L} \frac{1}{\epsilon_0} Q_{enc}$

Now for $r<a$ $Q_{enc}=0$

while for $a<r<b$

$$Q_{enc} = \int \rho \, dV = \int_a^r ds \int_0^{2\pi} s \, d\theta \int_0^L dz \, \rho(s)$$

$$= \int_a^r ds \; s \, 2\pi L \, \rho(s)$$

$$= \int_a^r ds \; 2\pi L \frac{k}{s}$$

$$= 2\pi k L \left[\ln(r) - \ln(a) \right]$$

$$= 2\pi k L \ln(r/a)$$

for $r>a$

$$Q_{enc} = 2\pi k L \ln(b/a)$$

So

$$E = \begin{cases} 0 & ; \quad r<a \\ \frac{k}{\epsilon_0 r} \ln(r/a) & ; \quad a<r<b \\ \frac{k}{\epsilon_0 r} \ln(b/a) & ; \quad b<r \end{cases}$$

P1.2

First we note the vector identity for a function ψ and constant vector $\mathbb{C}$ that

$$\nabla \times (\mathbb{C}\psi) = \nabla\psi \times \mathbb{C}$$

So we need to compute

$$\nabla \cos(f) \quad \text{with} \quad f = (\mathbb{k}\cdot\mathbb{r} - \omega t + \varphi)$$

$$= -\sin(f)\frac{\partial f}{\partial x}\hat{x} - \sin(f)\frac{\partial f}{\partial y}\hat{y} - \sin(f)\frac{\partial f}{\partial z}\hat{z}$$

$$= -\sin(f)\left[k_x\hat{x} + k_y\hat{y} + k_z\hat{z}\right]$$

$$= -\sin(f)\,\mathbb{k}$$

So we have

$$\nabla \times \mathbb{E} = \overbrace{\mathbb{E}_0}^{\mathbb{C}}\,\overbrace{\cos(\mathbb{k}\cdot\mathbb{r} - \omega t + \varphi)}^{\psi}$$

$$= \underbrace{-\sin(f)\,\mathbb{k}}_{\nabla\psi} \times \underbrace{\mathbb{E}_0}_{\mathbb{C}}$$

Now for the RHS $-\frac{\partial \mathbb{B}}{\partial t}$

$$-\frac{\partial \mathbb{B}}{\partial t} = -\frac{\partial}{\partial t}\left[\frac{\mathbb{k}\times\mathbb{E}_0}{\omega}\cos(f)\right]$$

$$= -\frac{\mathbb{k}\times\mathbb{E}_0}{\omega}(-\sin(f))\frac{\partial f}{\partial t}$$

$$= -\frac{\mathbb{k}\times\mathbb{E}_0}{\omega}(-\sin(f))(-\omega)$$

$$= -\mathbb{k}\times\mathbb{E}_0\sin(f)$$

So

$$\nabla\times\mathbb{E} = -\mathbb{k}\times\mathbb{E}_0\sin(f) = -\frac{\partial \mathbb{B}}{\partial t}$$

Vector Identity

$$\nabla\times(\mathbb{C}\,\psi)$$

$$= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\times(\mathbb{C}\,\psi)$$

$$= \left(\frac{\partial \psi}{\partial x}\hat{x} + \frac{\partial \psi}{\partial y}\hat{y} + \frac{\partial \psi}{\partial z}\hat{z}\right)\times\mathbb{C}$$

$$= \nabla\psi\times\mathbb{C}$$

P1.3 The loop is a circle of radius r and the field is at all places parallel to this loop so $\oint \mathbf{B} \cdot d\mathbf{l} = \int B \, dl$

also the field is uniform around the loop so $\int B \, dl = B \int dl = B 2\pi r$

Thus $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$

Becomes $B = \frac{\mu_0}{2\pi r} I_{enc}$

for $r < a$ $I_{enc} = 0$

for $a < r < b$

$$I_{enc} = \int \mathbf{J} \cdot d\mathbf{A} = \int J \, dA = \int_a^r ds \int_0^{2\pi} s \, d\theta \, \frac{k}{s}$$

$$= 2\pi k (r-a)$$

$$B = \mu_0 k \begin{cases} 0 & \text{for } r < a \\ (r-a)/r & \text{for } a < r < b \\ (b-a)/r & \text{for } b < r \end{cases}$$

P1.4 (a)

$$\nabla \cdot (\mathbb{C}\psi) = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \mathbb{C}\psi$$

$$= \frac{\partial \psi}{\partial x} C_x + \frac{\partial \psi}{\partial y} C_y + \frac{\partial \psi}{\partial z} C_z$$

$$= \nabla \psi \cdot \mathbb{C}$$

So with $\mathbb{C} = \mathbb{E}_0$ and $\psi = \cos(f)$

$$\nabla \cdot (\mathbb{E}_0 \cos(f)) = (-\sin(f)\,\mathbb{k}) \cdot \mathbb{E}_0$$

But $\mathbb{k} \perp \mathbb{E}_0 \Rightarrow \mathbb{k} \cdot \mathbb{E}_0 = 0$ so

$$\nabla \cdot \mathbb{E} = 0$$

Thus we need that $\rho = 0$.

(b)
In the same way

$$\nabla \cdot \mathbb{B} = (-\sin(f)\,\mathbb{k}) \cdot \frac{\mathbb{k} \times \mathbb{E}_0}{\omega} \cos(f)$$

But $\mathbb{k} \cdot (\mathbb{k} \times \mathbb{E}_0) = 0$ so $\nabla \cdot \mathbb{B} = 0$

(c) $\nabla \times \mathbb{B}$ is also of the form $\nabla \times (\mathbb{C}\,\psi)$

with same ψ and $\mathbb{C} = \frac{\mathbb{k} \times \mathbb{E}_0}{\omega}$

so $$\nabla \times B = -\sin(f)\, \mathbb{k} \times \frac{\mathbb{k} \times \mathbb{E}_0}{\omega}$$

$$= \sin(f)\, \frac{k^2}{\omega}\, \mathbb{E}_0$$

while
$$\mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t} = \mu_0 \epsilon_0\, \omega\, \mathbb{E}_0 \sin(f)$$

and $\nabla \times \mathbb{B} = \mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t}$ if

$$\mu_0 \epsilon_0\, \omega = \frac{k^2}{\omega}$$

or $$\frac{\omega}{k} = \sqrt{\frac{1}{\mu_0 \epsilon_0}} = c$$

This satisfies eq. 1.4 if $\mathbb{J} = 0$

P1.5

Memorized!